DOCUMENT RESUME

ED 175 706 SE 028 694

AUTHOR	Luce, R. D.
TITLE	Studies in Mathematics, Volume I: Some Basic Mathematical Concepts.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	59
NOTE	169p.
EDRS PRICE	MF01/PC07 Plus Postage.
DESCRIPTORS	Curriculum; *Games; Independent Study; *Inservice Education; *Instruction; Mathematics; *Mathematics Education; Secondary Education; Secondary School Mathematics; *Set Theory
IDENTIFIERS	*Functions (Mathematics); *School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG books on various topics directly related to high school mathematics courses, designed for the benefit of those mathematics teachers who wish to improve their teaching through independent reading. Particular attention is paid to topics which play an important part in the courses developed by SMSG. Chapter topics include sets, relations, orderings, functions, axiomatization of functions, and axiomatization of mathematical games. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED175706

U S DEPARTMENT OF HEALTH
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

SMSG

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

STUDIES IN MATHEMATICS
VOLUME I

Some Basic Mathematical Concepts

[illegible]

SE 028 694

SCHOOL
MATHEMATICS
STUDY GROUP

STUDIES IN MATHEMATICS

VOLUME I

Some Basic Mathematical Concepts

by R. D. LUCE

Prepared for the GENERAL ELECTRIC COMPANY
Reproduced and distributed by the SCHOOL MATHEMATICS STUDY GROUP
Under a grant from the NATIONAL SCIENCE FOUNDATION

Copyright 1959 by the General Electric Company
Printed in the U.S.A.

FOREWORD

For the benefit of those mathematics teachers who wish to improve their teaching through independent reading, the School Mathematics Study Group plans a series, STUDIES IN MATHEMATICS, on various topics directly related to high school mathematics courses. Particular attention will be paid to topics which play an important part in the courses being developed by the School Mathematics Study Group.

One such topic is elementary set theory. Indeed, this plays an important role in practically all of the recent recommendations for the improvement of high school mathematics courses. We are indeed fortunate to obtain, for the first volume of the STUDIES IN MATHEMATICS series, an extensive exposition of the basic concepts of elementary set theory together with illustrations of the use of set concepts in various parts of mathematics.

This material was prepared by Professor R. D. Luce, of Harvard University , for a teaching program of the Operations Research and Synthesis Consulting Service of the General Electric Company. The School Mathematics Study Group is grateful to the General Electric Company for permission to make this material available to high school teachers.

Although some revisions and corrections have been made, this is essentially the first draft prepared by Professor Luce. It is hoped that a revised and extended version of this material will be prepared in the future.

TABLE OF CONTENTS

Page

INTRODUCTION 1

CHAPTER

I. SETS 7

Introduction 7
Membership and Notation 8
Subsets 12
A Paradox 17
Union, Intersection, and Complement 19
Set Functions 32
Algebras of Sets 34
Legislative Schemes 37

II. RELATIONS, ORDERINGS, AND FUNCTIONS 47

Product Sets 47
Relations 49
Three Important Special Properties 51
Equivalence Relations 53
Matrix and Graphical Representations of Relations . . . 55
Games in Extensive Form 63
Orderings 68
Functions 74
Summary of Remarks 79

III. AXIOMATIZATION OF FUNCTIONS 81

Introduction 81
Defining Functions 82
Some Well Known Properties of Ordinary Functions . . . 86
Differential Equations 93
Axiomatization and Inconsistency 98
The Arrow Social Choice Problem 101
Consistency and Uniqueness 111
The Information Measure 115
The Shapley Value of a Game 121
Non-Uniqueness 126
The Non-Uniqueness of Linear Utility Functions 128

CHAPTER Page

IV. AXIOMATIZATION OF MATHEMATICAL GAMES 139

Introduction 139
Some Terminology 143
Categoricalness and Isomorphism 146
Boolean Algebras 151
Switching Circuits 157
Existence of Linear Utility Functions 162

SOME BASIC MATHEMATICAL CONCEPTS

INTRODUCTION

As you first study this material, it may seem both more fun and much less useful than some other mathematical topics you have studied or are studying. Unlike the other subjects you are studying, most likely you will never be able to quote a specific result from these materials which you have used directly to solve a problem, but if you absorb the ideas you will find yourself formulating problems and reasoning about them in a way that is new and useful.

We shall be concerned with some of the basic building blocks of precise, logical thought -- a whole collection of ideas and concepts and methods which, in themselves, are simple and almost familiar, yet which can be pyramided and interwoven to yield subtle theories of considerable power and depth. These will be qualitative ideas, and so the resulting mathematics is vastly different from the quantitative subjects with which you are more familiar: arithmetic, algebra, and the calculus. Because the notions are qualitative, they are far more related to much of our ordinary language and ways of thinking -- to classifying, relating, and ordering concepts and things -- than they are to numerical ideas. A thorough understanding of these basic elements of modern mathematical thought permits one to reason far more clearly about complex qualitative situations, just as a good knowledge of the number system permits one to reason (often in the form of a calculation) effectively about quantitative problems.

Looked at another way, we shall be concerned with things discrete, whereas classical mathematics deals with continuous phenomena. The mathematics of physics, which received its primary impetus from Newton and Leibnitz and which flowered in the hands of the great eighteenth and ninteenth century mathematicians, allows us to formulate and to solve problems involving such concepts as length, weight, time, etc.

The common property of these concepts is that they are infinitely divisible. For such problems the traditional mathematics seems ideally suited, and one cannot expect much, if any, new understanding of them from the mathematics we shall study here. But in other areas of applications -- including some microscopic physics -- the principle of infinite divisibility is simply not valid, not even as a plausible first approximation. Atoms exist in a small finite number of states. Half a horse is no horse at all. General Electric's output of large industrial turbines is not any real number, but a rather small integer. Being on a particular committee in a company may be a relevant fact for some problems, but this hardly seems a numerical concept. What is needed is a discrete mathematics to parallel and supplement our better known continuous mathematics.

It is well to keep in mind that much of the world of immediate perception is discrete, and if one point of view must be held less strange than the other, it is not evident that the continuous-quantitative is it. A good deal of training, such as four years of engineering school, is needed before a person will concentrate his attention only on variables having a continuous character. Yet, you may ask: if the discrete view of the world is quite natural and if we have an appropriate mathematics to deal with it, why then is the child not taught this as well as the traditional fields? Many would say because it is too difficult, too advanced, and of too little use -- and their arguments seem strong. For example, there is its history of development. The ideas we shall speak of came into being only toward the end of the last century -- thousands of years after Euclid froze elementary geometry and hundreds after the calculus was invented. No one will deny that, once George Boole (1815-1864) and G. Cantor (1845-1918) put forth these new concepts, they were accepted as fundamental to all mathematical thought and that mathematics (including that of continuous processes) was revolutionized; but many will point out how recent this development has been. Can a child be expected to learn what the mathematical community could not develop until 50 years ago? Some think yes.

The ideas are simple. They are so simple, so immediate, so much a part of our language and thinking that is hard to realize that there might be a purpose served in abstracting them into a symbolic system. Yet, as you will see, once the abstractions are pointed out, they are clear and understandable -- in fact, you will often have the feeling that you knew about this or that all along, but had just never troubled to think it out clearly. And you will be right. Were the abstracting the end of it, it would be banal; but that it is not, for once the formalization is effected it assumes a life and power of its own which is far richer than will first seem possible.

The logical and clearest development of a sequence of ideas is rarely, if ever, the historical one. Complex problems are recognized, tackled, and often solved before the more basic and, retrospectively, simpler problems are seen and resolved.

In these notes we shall try to do three things. First, we hope to give you some idea of the concepts which are available and where you can find out more about them. We will draw heavily on your unformulated experience in such matters and we shall not attempt to be as careful about the niceties as would be necessary in a course in pure mathematics. Second, we will delve into one collection of ideas which is particularly useful, attempting to show reasonably explicitly how a modern mathematician thinks and works with such qualitative problems and what kinds of deductive steps are involved. This work will be somewhat more formal and may seem a bit taxing, but we will try to build up to it fairly gradually. As much as anything, your difficulties will arise from our ordinary habit of reading into statements more or less than they imply. One of the qualities of mathematics -- often, though not always, a virtue -- is precision. Professionally, mathematicians pride themselves on saying exactly what they mean and meaning exactly what they say. For this reason, statements have to be taken literally -- a curiously difficult thing to do, especially when one has an inadequate intuitive grasp of what is being done. Third, we will present several examples from the social sciences where interesting results have been obtained by welding together a number of these almost trite ideas. Here you will see some of the intellectual power which arises from an adroit interweaving of the almost trivial. It is an impressive feat once it is understood.

In all of this we will not stress manipulative skills; there is not time for that. We will feel successful if you gain some idea of what the concepts are, what they might be good for, and where more can be found about them. For those wanting to learn more of the details a limited number of general references will be scattered about the notes. In addition, the following books are basic references to the whole area:

Kemeny, J. G., Snell, J. L., and Thompson, G. L., Introduction to Finite Mathematics, Prentice Hall, in press (probable date of publication: January 1957).

Kershner, R. B. and Wilcox, L. R., The Anatomy of Mathematics, Ronald Press, New York (1950).

Stabler, E. R., An Introduction to Mathematical Thought, Addison-Wesley, Cambridge (1953).

Wilder, R. L., Introduction to the Foundations of Mathematics, Wiley, New York (1952).

Two basic texts of a more advanced character are:

Birkhoff, G., Lattice Theory, American Mathematical Society, New York (1948).

Birkhoff, G. and MacLane, S., A Survey of Modern Algebra, MacMillan, New York (1946).

While we cannot stress manipulations, here, as in all rigorous disciplines, a real understanding does not usually result from just reading and listening. The real import of the ideas and the fine shadings of meaning do not come across until you try to work with them, which means doing problems. For the most part we have chosen very simple ones, but they won't seem so until the ideas are clearly grasped. These should be worked even when you feel that you have understood perfectly what you have read. If you have understood, the cost in time of doing them will be slight; and if you haven't, the gain will be great.

The starred sections and the passages in small print are a little more difficult than the rest of the material. They may be omitted if one chooses without causing later difficulties, for the rest of the text is self contained without them. However, it is recommended that they be read, possibly on a second time around, for it is in these sections that we go beyond the more elementary concepts and attempt to show how something can be done with them.

CHAPTER I

SETS

1.1 INTRODUCTION

One of the simplest and most ubiquitous of mental operations is recognition -- deciding whether an object of perception does or does not possess certain characteristics and whether along some dimension one object is the same as or different from another. Set theory uses this as its starting point. One could doubt that such a trivial base can lead to anything much in particular; and, yet, from it and the rules of logic one can derive the whole of contemporary mathematics. Of course, we will not attempt to do so here; we will only try to give some of the elements of set theory and a few of the applications which arise from it. In practice, these are the useful things to know, for no mathematician traces back his work to the most fundamental formulation; it is enough to know (or imagine) that it can be done.

Analogous situations exist in physics. It was important to show that from the microscopic theory of gases, kinetic theory, it is possible to derive macroscopic thermodynamics, but it would hardly make sense for an engineer designing a heat exchanger to return to molecular principles. In the same way, it is completely uneconomical to do much of every day mathematics by returning to set theory -- certainly this is true of most engineering and physics mathematics. And so most engineers and physicists are not taught set theory. But, at present, when human behavior is involved, most mathematical analyses do begin in a very basic way. Possibly in the future an elaborate superstructure will be constructed for the behavioral sciences and it will again be impractical to return to first principles, but this is not yet so.

1.2 MEMBERSHIP AND NOTATION

The concept of a set, or class, will be accepted as intuitively known, or, as one says in mathematics, it is undefined. Though we shall not attempt to analyze its meaning into more primitive terms, it would be unkind not to attempt to aid the intuition by examples and suggestive discussion. A set separates the universe into two parts: those things in the set and those not in the set. Put another way, a set is determined by a rule or property: those objects of perception which satisfy the rule belong to the set, those which do not, do not belong to it. Consider the property of being human and exceeding six feet in height. This defines a set which we might call the set of "tall people." Given any object whatsoever, one must decide, first, whether it is a human being and, if so, whether that person is taller than six feet. There are two acts of recognition required for each object.

The rule defining a set can be almost anything, however weird, provided it meets one important condition. It must be possible to decide for any object whether or not it satisfies the rule. The set defined by such a rule is sometimes (redundantly) described as "well defined." In a great deal of mathematics this stipulation is taken pretty much for granted without explicit discussion; but it must never be forgotten that it is a very stringent requirement, one not so easily met in applications, particularly when people are involved. A person is confronted with a multiple choice question, i.e., one having a well defined set of alternative answers. There is no problem in stating the set of answers offered to him, or the answer he gives, but what of the set of alternatives he actually considers before making his choice? In judging his performance this may be crucial. You know this much: it must include the answer he actually chooses and it is bounded by the available set of alternatives, but it could happen that he does not consider all of them. Is it possible to decide whether a given alternative is or is not in the set he considers? No really effective way is now known to ascertain this, but that does not mean this will always be so. We offer this example only as a warning that it is easy in practice to suppose tacitly that a set is well defined, when in fact it isn't.

For sets with a finite number of elements, the simplest, and invariably unambigious, rule is to list all the elements in the set. For example, the set consisting of the three integers "one," "two," and "eight" can be specified by listing them as {1,2,8}. A more bizarre set having three elements is {this piece of paper, Queen Elizabeth's coronation crown, the sun}. In each case, the order of writing the several elements is immaterial, thus

{1,2,8}, {1,8,2}, {8,1,2}, {8,2,1}, {2,8,1}, {2,1,8}

all denote the same set.

Whenever one explicitly lists all the elements of a set, it is conventional to surround the symbols for the elements by curly brackets as we have done.

More generally, a set is characterized by some property possessed by its elements and not possessed by any other objects. Such a rule must always be used when describing a set having an infinite number of elements. For example, the set consisting of all numbers greater than zero is called the "right half line." (The term arises from the geometrical representation of all numbers by a line.) The property which specifies the elements in the set is "being a real number and being greater than 0." Thus, 1, π, 1,036.24, etc. are in the set; 0, -1, -10^6, etc. are not. "The members of the Senate of the United States on January 1, 1956" defines a set of 96 people whom one can list. "The President of the United States in 1942" singles just one person,* and it affords an example of a set consisting of a single element. One must distinguish between a set having but one element and the element itself. Congress observes this nicety when, on occasion, it by-passes the rule that it cannot write a law which names and fires a person. This it does by abolishing all jobs satisfying certain characteristics, so choosing the characteristics that there is a single position -- held by the man they wish to get -- satisfying them. Technically, Congress has written the law in terms of a set, not an element. Notationally, if a denotes an element, then {a} denotes the set consisting of just that element.

---

*Had we changed the date, say to 1945, this would not necessarily be the case.

Observe that in our discussion there have really been three central undefined concepts: set, element, and belongs to. These are related by a particular element either belonging to or not belonging to a given set. It is useful to be able to symbolize this primitive relationship briefly; it is done as follows: If a is an element of, i.e., belongs to, the set A, we write

$$a \in A.$$

In this context, the symbol $\in$ (Greek epsilon) can be read in a variety of ways: belongs to, is an element of or is a member of. Thus, 2 1,2,8 , i.e. 2 is a member of the set consisting of 1, 2, and 8. Sometimes we also want to say that "a is not a member of the set B," and this we symbolize by

$$a \notin B,$$

where the slash means "not." So we read $\notin$ as: does not belong to, is not an element of, or is not a member of.

We now have notational ways of representing two things. First, we can symbolize a finite set by explicitly listing its elements: {a,b,c,d}. Second, if we have symbols for a set and an element, we can symbolize that the element is or is not a member of the set: $a \in A$, $a \notin B$. It would also be useful to have a symbolic way to represent a set which is characterized by some rule or property, say property P.

It is conventional to denote (and that is all it is - a name) this set by:

$$\{x \mid x \text{ has property } P\}.$$

In this notation, x is a generic element of the set being defined by the property P, and the vertical bar is read "such that." Thus, we read the symbol as "the set of all elements x such that x has property P." For example, the right half line mentioned above would be presented as:

$$\{x \mid x \text{ is a real number}, x > 0\}.$$

Two conventions we have employed had best be made explicit. In so far as possible, capital Latin letters will be used to denote sets and small ones to denote their elements, and the generic element of a given set will often be symbolized by the same letter as the set, such as $a \in A$. We will not always be able to hold to these conventions since sometimes sets are elements of other sets and a particular letter may not be suitable for use as an element, but to the extent that we can we will follow them.

A variety of synonyms exist for the word set. The most common, and with some authors the preferred term, is class. Also used are aggregate, collection, and family. There are some implicit conventions as to when each is used, but we will not go into that here except to say that one usually speaks of a class or collection of sets, rather than a set of sets.

Problems

1-In two different ways, present symbolically the set of positive integers which divide evenly into 12.

2-Display the set $\{x \mid \text{is an integer}, x^2 = x\}$ in another way.

3-Devise a property P which is meaningful for all integers, similar to the one in problem 2, which allows the set {0} to be displayed in the form {x | x has property P}. Do the same thing for the set {1}.

1.3 SUBSETS

In a certain and somewhat facetious sense, one can characterize much of modern mathematics as the generation of new sets from old. There is, of course, much more to it than this, but constructing new sets having special properties is always going on. As we proceed you will see how this can be done with profit. Our first example of it is the formation of subsets. For example, the set of all Republican Senators is a subset of the set of all Senators. The executive personnel of General Electric is a subset of its employees. The set of transformers produced last week by General Electric is a subset of all its products for that week, and also a subset of all its products for all time past, and also a subset of all transformers produced during the year, etc.

Formally, if A and B are each sets, A is a subset of B if every element of A is also an element of B. In each example you can see that this is the case: a General Electric executive is also a General Electric employee, etc. Of course, the converse is not generally true -- there are still employees who are not executives.

If A is a subset of B, we then also say that B is a superset of A; however, this term will be used much less often than "subset."

Certain subsets are especially distinguished. Every set is a subset of itself, for if A is a set and $a \in A$ then, repeating ourselves, $a \in A$. Often, we want to exclude this trivial case when talking of subsets, and so we need a term to refer to subsets of a set which are different from the set itself. The term used is proper subset.

Suppose $a \in A$, then {a} is a subset of A. That is, each of the single element sets formed from the elements of A is a subset of A.

One of the most useful sets, though at first it seems senseless, if not silly, is the one (it can be shown to be unique) which has no elements; it is called the empty or null set. The major reason for introducing this apparently vacuous concept is this: you may set up a certain property and discuss the set of elements having this property, only later to discover that there were no elements satisfying it. It is more convenient not to have to deal with this vacuous case any differently from more substantial sets. The set of all United States Senators under 25 years of age is an example. But, you will say, no one would ever consider this set, for it is clearly empty. Although that is true for this example, there are other cases where the emptiness of the defining characteristic is not nearly so evident. If you don't know much about cats, to speak of tri-colored male cats does not seem unreasonable. Later we shall come to other reasons for introducing the null set.

Notationally, we shall denote the null set by $\emptyset$.

We observe that the empty set is a subset of every set, for every element of $\emptyset$ is, by its non-existence, also an element of every other set.

Once again, it will be convenient to have a short symbolism for discussing subsets. For the phrase "A is a subset of B" we will write $A \subset B$. This symbolism is not just accidently similar to the "less than" sign, $<$, for numbers, for if A is a subset of B it is, in a sense, "less than" B. However, one never says that, rather that A is a subset of B or that A is included in B. The mark $\subset$ is known as the inclusion symbol.

Actually, what we have been doing just now is making a definition, so let us summarize it formally:

Definition: If A and B are sets, we say A is a subset of B, and write $A \subset B$, if $a \in A$ implies $a \in B$.

In terms of the inclusion symbol, we always have:

$A \subset A$; $\emptyset \subset A$; and if $a \in A$, then $\{a\} \subset A$.

Not only will we want to talk of specific subsets of a set, but also of the set of all subsets of a given set. For this too we want a symbol. We could simply say that we will write 2^A for the set of subsets of A, but it will be easier to remember this if we suggest how it arose. Suppose that A is a finite set with the n elements $a_1, a_2, \ldots, a_n$ (the numbering of the subscripts does not matter except that we hold it fixed and that different elements have different subscripts). One way of denoting a subset of A is as follows: We ask if the first element, a_1, is present in the subset or not. If it is we write down a 1, if not a 0. Then we ask if the second one is in the subset, and we write a 1 to the right of our first number if it is, a 0 if it isn't. We continue the process through all n elements, and in this way we get a sequence of n numbers, each either 0 or 1, which describes the subset. For example, suppose $n = 4$, then the subset $\{a_2, a_4\}$ is symbolized by 0101. Similarly, the single element set $\{a_3\}$ is represented by 0010. We also note that any such sequence of 0's and 1's represents a subset, i.e., the binary numbers of length n correspond perfectly with the subsets of any finite set of n elements. But it is easy to see that there are 2^n such numbers, for there are two choices, 0 or 1, for each place and there are n places. The symbol, then, for the set of subsets of A, 2^A, is suggested by the number 2^n. This symbol is used even if A has an infinity of elements, where the binary representation of the subsets might break down.

Formally,

$$2^A = \{B \mid B \subset A\}.$$

As an example, suppose $A = \{a,b,c\}$, then

$$2^A = \{A, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}, \emptyset\}.$$

The distinction between the symbols $\in$ and $\subset$ must be kept in mind, for they are sometimes interchanged by novices. One can never be directly substituted for the other, for $\in$ establishes a relation between an element and a set, whereas $\subset$ relates a set to a set. A few examples will make this clear: If $B \subset A$, then $B \in 2^A$. If $a \in A$, then $\{a\} \subset A$ and $\{a\} \in 2^A$.

In set theory, just [illegible] in ordinary algebra, it is quite possible to specify the same thing in several different ways; sometimes this is done on purpose, sometimes inadvertently. In algebra, we may introduce two apparently different numbers x and y, but by some chain of reasoning come to the conclusion that they are the same number, in which case we write $x = y$. Similarly, with sets we may think we are defining two different sets A and B only later to find that they have the same elements -- that they are identical. In that case it seems appropriate to extend the use of the equal sign and to write $A = B$. When you see such an expression and A and B are sets, you must remember that numbers are not involved at all; it simply means that the two sets A and B have exactly the same elements, not just the same number of elements. Example: suppose A is defined to be the set consisting of the final four words in the last sentence, and B is defined to be {elements, number, same, of} , then $A = B$. Less trivial example: Let A be defined to be the set of two people comprising the Democratic nominees for President and Vice President in 1940 and B the President and Vice President in 1941, then it is known as an historical fact that $A = B$, though it could have happened otherwise.

Note that if $A = B$, then A is included in B and B in A; and, conversely, if both inclusion relations hold, the two sets must have the same elements. This gives us an easy way formally to define equality:

Definition: If A and B are sets, we say they are equal, and write $A = B$, if both $A \subset B$ and $B \subset A$.

This concept of equality has all the properties one usually associates with equality among numbers, namely:

$A = A$	(reflexive)
if $A = B$, then $B = A$	(symmetric)
if $A = B$ and $B = C$, then $A = C$	(transitive)

The terms in parantheses are standard for these three properties, and they will be discussed more fully in Chapter 2. One can immediately establish that these properties hold from the fact that inclusion is reflexive and transitive :

$$A \subset C$$

$$\text{if } A \subset B \text{ and } B \subset C, \text{ then } A \subset C.$$

(The reader should show this). These last two properties can be shown from the definition of inclusion. We have previously discussed and established the first of these. Consider the second: Suppose $a \in A$. Since $A \subset B$, we know from the definition of inclusion that $a \in B$. But since $B \subset C$, the same definition implies that $a \in C$. But if $a \in A$ implies $a \in C$, then $A \subset C$ by definition.

In any particular discussion or problem, we will always restrict ourselves in advance as to what elements we want to talk about. In other words, we will always specify a universal set U at the start which is chosen to include everything we shall want to mention. It may seem strange to bound ourselves by such a convention, but in practice it is an extremely useful device and it will also help us to avoid certain logical difficulties (see next section). The practical merit amounts to this: if you don't tell your listener what you are talking about, he literally won't know except to the extent he can infer it from your statements about the unknown universe of discourse.

When we are dealing with basic ideas, this indirect method will be used -- it is known as the axiomatic method. But when we are discussing known elements and sets, it is important to specify which ones. Amateurs often fail to be explicit on this score, and it can be very tricky indeed to decipher their later marks and symbols.

Problems

1-Let A denote the employees of a company, W the set of female employees, E the set of executives, p the president, j a particular male janitor. State all relationships you can think of using ϵ and $\subset$.

2-Write out all the subsets of {janitor, president, set of women employees}.

3-If A is a set having 3 elements, how many elements are there in the set of subsets of 2^A (which we shall denote by 2^{2^A})? If A has n elements, how many are there in 2^{2^A}? How would you denote the elements in 2^{2^A} which are formed from single element sets of 2^A?

4-Let A and B be finite sets. Show $2^A \subset 2^B$ if and only if $A \subset B$.

1.4 A PARADOX

The logical difficulties in set theory are famous, mainly because they were so profoundly shocking to the mathematical community. Some years after set theory was first introduced and when it was already being widely used throughout mathematics, it was discovered that, by using simple reasoning of a type generally employed in mathematical arguments, deep inconsistencies could be exhibited. Since these arguments do not differ from those used in everyday mathematics, which has had such rich and useful conclusions, much unease was generated. And while a good deal of work has since occurred in the foundations of mathematics, it cannot yet be said that all is well. While we cannot go into this work, it is easy to exhibit one of the paradoxes.

The one we shall describe is known as the Russell paradox, named after its famed author Bertrand Russell. As we have seen (e.g., the set of all subsets of a given set) the elements of sets may themselves be sets. Thus, a priori, there is the possibility that there is at least one set which is an element of itself! While we have not exhibited such a set, it is conceivable that one exists. In any case, let us call any set not having this property, i.e., any set not having itself as an element, an ordinary set. These we know do exist. Let W denote the set of all ordinary sets. Question: is W itself ordinary?

One way to show that W is ordinary is to assume the contrary is true and to show that this leads to a contradiction, i.e., to show the assumption that W is not ordinary leads to an absurdity. This we do. If W is not ordinary, then by definition W is an element of itself. But, by choice, all of the elements of W are ordinary sets, and so we have a contradiction. Thus, we must conclude that the supposition is false and that W is ordinary.

This seems fine. But suppose we had begun on the other tack of trying to show that W is not ordinary by assuming it is ordinary and arriving at a contradiction. If W is ordinary, then W is not an element of itself according to the definition of an ordinary set. But all ordinary sets are by choice included in W, and again we have a contradiction. Thus, we must conclude that W is not ordinary.

The dilemma is clear: by well accepted deductive procedures we have proved both that W is ordinary and not ordinary. The resolution is to try, in one plausible way or another, to exclude W and other objects like it from being classed as the same sort of sets as the ones which are its elements. This we cannot go into.

Two closely related paradoxes which are easily remembered are these: Consider the assertion " This sentence is false." If you assume it is true, then you can conclude it is false; if you assume it false, you can conclude it true. Consider the barber in a town who shaves everyone who does not shave himself. Who shaves the barber?

For more discussion of the problems lying at the foundations of mathematics, see Wilder's book mentioned in the Introduction.

1.5 UNION, INTERSECTION, AND COMPLEMENT

To the child and the mathematician there are certain natural operations which can be carried out with sets. For the rest of us these conceptual operations seem slightly illegal, for they are carefully outlawed during early schooling when the child is first introduced to another basic notion, that of number. It is obvious to the child that he can "add" a set of books that he has to a set of pencils he uses, for that is exactly what he does when he places them together in a bag to carry them home. Sometimes he wants to treat the two sets as separate, other times as a unit. It depends upon his purpose. This certainly is not the addition to which the teacher is addressing herself when she tells him that he can only add "likes to likes" and that when he adds likes they must not overlap. The difficulty she is trying to avoid can be seen clearly by considering one set consisting of m books and n pencils and another consisting of the same n pencils and p pads of paper. The "logical sum" consists of the set of books, pencils, and pads, and it has $m + n + p$ elements in it, whereas the simple arithmetic sum of the numbers of elements of the two sets gives the number
$(m + n) + (n + p) = m + 2n + p$.

Equally well, there is for sets something somewhat analogous to multiplication, namely, the set of elements which are common to two given sets. In the above example, the n pencils are in common. There are people who are wealthy and others who are smart, and those who are in common to the two sets are both wealthy and smart.

Suppose, then, that U is our universal set and A and B are two of its subsets. We want to be able to symbolize the set of elements which are either in A or in B or in both -- the analogue of addition. There are several symbols in use, the most common being $A \cup B$ and $A + B$. We shall use the former to avoid confusion with numerical addition. So we make the following

Definition: Let U be given and $A, B \subset U$, then

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

We speak of $A \cup B$ as the union of A and B, or as A union B. The term logical sum is also widely used, but we will avoid it.

Similarly, we want to denote the set of elements common to A and B, so we make the following

Definition: Let U be given and $A, B \subset U$, then

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

We speak of $A \cap B$ as the intersection of A and B, or as A intersect B. The term logical product is also used.

If one writes $A + B$ instead of $A \cup B$, then it is customary to write AB instead of $A \cap B$.

If in the above example we let A = {books, pencils} and B = {pencils, pads}, then

$$A \cup B = \{\text{books, pencils, pads}\}$$
$$A \cap B = \{\text{pencils}\}.$$

There is a very useful graphical device, known as a Venn diagram, for thinking about these and more complex relations among subsets.

Whatever our sets may be, finite or infinite, we represent them in a loose analogy by regions in the plane. Thus, we first select an arbitrary region such as in Fig. 1 to represent the universal set U. Then we introduce subregions to represent our subsets, sometimes shading them for greater clarity. Thus, in Fig. 2 the whole shaded area represents A ∪ B, while in Fig. 3 the shaded area is A ∩ B.

Fig. 1

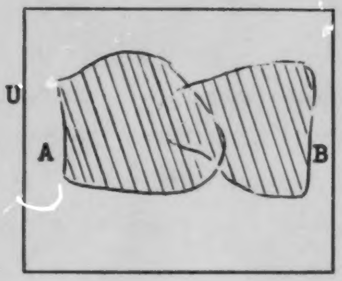


Fig. 2

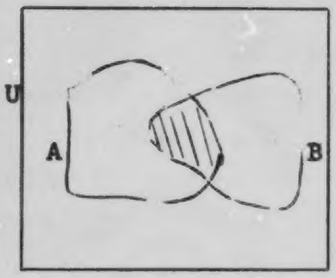


Fig. 3

Such diagrams can suggest a variety of questions. For example, consider the subsets indicated in Fig. 4. What about A ∩ B? It is clear that there is no common region to shade, that they have no elements in common. In our previous terminology, A ∩ B is the empty set. Here seems to be a case where the empty set comes in handy, for it is nice always to think of A ∩ B as being a set, just as A ∪ B is always a set. As you might expect, we often have to distinguish whether or not A ∩ B = ∅ -- much as in ordinary algebra we have to specify whether a number if different from zero or not. For this reason a special term is introduced, namely: A and B are called disjoint if A ∩ B = ∅.

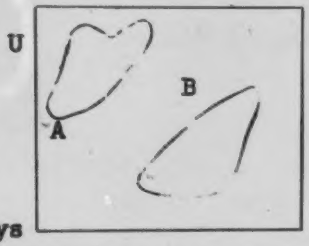


Fig. 4

An example may be illustrative. Suppose we take as our universal set a list of corporate functions which must be considered some executive's responsibility. Let each executive in a particular company list those functions for which he considers himself responsible. Thus, each executive specifies a subset of the given universal set. It might be considered desirable for each pair of these subsets to be disjoint, since otherwise there would be overlapping responsibilities. Of course, the example also suggests that there should be at least one executive responsible for each function. That is, if we extend our idea of union beyond just two subsets (see below), the union of all these subsets should be the universal set. A set of subsets having these two properties - every pair of subsets is disjoint and every element of the universal set is in one of the subsets - is known as a partition of the universal set. This is a useful concept and it will arise later in another context.

Another idea suggested by the Venn diagram is shown in Fig. 5. Here we have shaded everything in U which is not in the given subset A. This set, known as the complement of A (with respect to U) will be symbolized by $\bar{A}$. Formally,

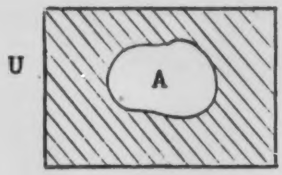


Fig. 5

Definition: Let U be given and $A \subset U$, then

$$\bar{A} = \{x \mid x \in U \text{ and } x \notin A\}.$$

In this concept the role of the universal set is vital. Frequently we will simply speak of the complement of a set, but it will always be with implicit reference to a particular universal set. As you can see, it is much more important to know the universal set when complements are mentioned than for unions and intersections, since in any universal set which includes both A and B, $A \cup B$ and $A \cap B$ always refer to unique sets but this is not true for complements.

The concept of complement can be generalized very easily to what is known as the difference between two sets. The difference between A and B, denoted by A-B, is simply the set of elements in A which are not in B, i.e., the set of elements common to A and the complement of of B. See Fig. 6. So we are led to

Definition: Let U be given and $A, B \subset U$, then

$$A - B = A \cap \overline{B}.$$

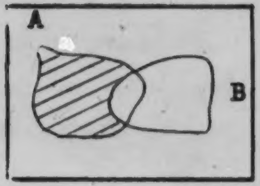


Fig. 6

Problems

1-In a Venn diagram with subsets A and B identify the following subsets:

$$A \cup \overline{B} \;;\; \overline{A} \cap B \;;\; \overline{A} \cap \overline{B}.$$

2-Express the following subsets of U in simpler terms

$$\overline{\phi} \;;\; \overline{U} \;;\; A - A \;;\; A - \phi \;;\; A \cap \overline{\phi} \;;\; \phi - A \;;\; A \cup \overline{U} \;;\; A - \overline{A} \;;\; A \cap \overline{U} \;.$$

1.6 OPERATIONS

Beginning with a universal set U and considering subsets we have now introduced a number of "operations," namely:

inclusion	$A \subset B$
union	$A \cup B$
intersection	$A \cap B$
complementation	$\overline{A}$
difference	$A - B$.

It is plausible that there must be some interrelations among these, just as in arithmetic there are relations among addition, multiplication, less than, etc. The kinds of numerical properties we have in mind are these:

$$x(y + z) = xy + xz; \quad xy = yx; \quad \text{if } x > 0 \text{ and } y < z, \text{ then } xy < xz.$$

The question now is what relations hold among the operations for subsets.

The easiest way to suggest some of them, and later to gain an understanding of the ones we shall state, is to see what happens on a Venn diagram. A couple of examples will do. Suppose, first, that $A \subset B$. Then, what about $\overline{A}$ and $\overline{B}$? On a Venn diagram, $A \subset B$ simply

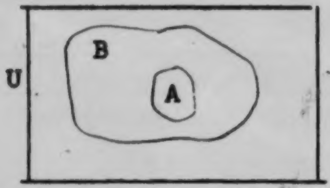


Fig. 7

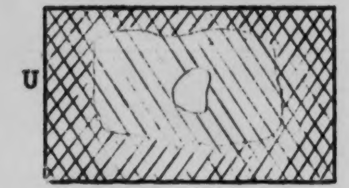


Fig. 8

means that the area representing A is included in the area representing B, as shown in Fig. 7. Now, shade in the areas representing the complements of A and B, as in Fig. 8, and we see that $\overline{B}$ is included in $\overline{A}$. Thus, the theorem we conjecture (it is by no means proved just because one special case holds) is: if $A \subset B$ then $\overline{B} \subset \overline{A}$. We will prove it in a bit.

As a second example, let's ask what happens when you take the complement of the union of two sets. In Fig. 9 the two sets A and B are indicated and the region $\overline{A \cup B}$ is shaded. Presumably, we want to express this as some operations involving A, B, $\overline{A}$, and $\overline{B}$, if possible. Let us draw A and B again, and shade in the areas representing $\overline{A}$ and $\overline{B}$,

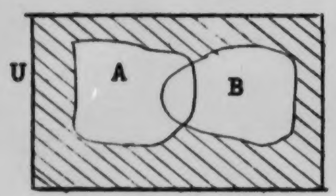


Fig. 9

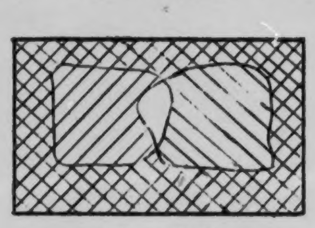


Fig. 10

as in Fig. 10. The area which has any shading at all represents $\overline{A} \cup \overline{B}$, and that clearly is different from $\overline{A \cup B}$. The area in which the shading is cross hatched represents $\overline{A} \cap \overline{B}$, and we see that is the same as the shaded area in Fig. 9. Thus, in this case, we conjecture: $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

We must stress that these diagrams only serve to suggest results; they do not constitute proofs. The danger in generalizing from a few special diagrams is either that we have managed to draw cases which possess certain peculiarities or that we have avoided one or two peculiar cases; in either event the conjecture will not hold in general. To get a true proof it is necessary to go back to the definitions of the operations in terms of elements in the sets and to verify that the two sides of an equality do in fact represent the same set. We shall prove these two conjectures as illustrations of the method of proof.

Theorem If $A \subset B$, then $\bar{B} \subset \bar{A}$.

Proof. By definition, $\bar{B} \subset \bar{A}$ if and only if $a \in \bar{B}$ implies $a \in \bar{A}$. To show this, we assume the contrary - namely, $a \in \bar{B}$ and $a \notin \bar{A}$ - and arrive at a contradiction. By definition of the complement, $a \notin \bar{A}$ implies $a \in A$. Since by hypothesis, $A \subset B$, we may conclude from the definition of inclusion that $a \in B$. But, by definition of the complement, this contradicts our assumption that $a \in \bar{B}$; thus, we must conclude that our tentative hypothesis $a \notin \bar{A}$ is false. Hence $\bar{B} \subset \bar{A}$.

In the mathematical literature, such a proof would either not be given -- the dangerous word "obvious" being written in its stead -- or would be given in much abbreviated form, e.g.: If $a \in \bar{B}$ and $a \in A$, then since $A \subset B$, $a \in B$, a contradiction; hence $\bar{B} \subset \bar{A}$.

Theorem $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Proof. By the definition of equality, the assertion is equivalent to the two inclusions,

$$\overline{A \cup B} \subset \bar{A} \cap \bar{B} \quad \text{and} \quad \bar{A} \cap \bar{B} \subset \overline{A \cup B}.$$

We prove each of these separately by supposing that we have an element in the set to the left of the inclusion relation and then show it must also

30

be in the set to the right. If $a \in \overline{A \cup B}$, then by definition of complementation, $a \notin A \cup B$. Thus, by definition of the union, $a \notin A$ and $a \notin B$, or putting this in terms of complements, $a \in \bar{A}$ and $a \in \bar{B}$. By definition of intersection, $a \in \bar{A} \cap \bar{B}$, so by definition of inclusion, the first inclusion relation is shown. To show the second, we suppose $a \in \bar{A} \cap \bar{B}$. Eliminating some of the steps, $a \notin A$ and $a \notin B$, so $a \notin A \cup B$, i.e., $a \in \overline{A \cup B}$. The theorem is proved.

It turns out that there are a vast number of such relationships among the several operations defined, each one of which can be proved in a manner similar to that just used. We do not propose to prove any more of these here; however, a few of the proofs will be given as exercises. Rather, we shall present without proof a small, selected set of true theorems. It is recommended that you draw the corresponding Venn diagram in each case, for only by examining each result individually will you become familiar with its content. This is necessary, for we will use them from time to time.

The arrangement of the theorems into horizontal groupings is both to make them easier to read and to indicate certain natural groupings. The partial parallel listing will be discussed later. The terms in parathenses to the right are standard in this area.

Let A, B, and C be any subset of a given universal set U.

1-	$A \subset A$			(reflexive)
2-	if $A \subset B$ and $B \subset C$, then $A \subset C$			(transitive)
3-	$\emptyset \subset A$	3'-	$A \subset U$	(universal bounds)
4-	$A \cup A = A$	4'-	$A \cap A = A$	(idempotent)
5-	$A \cup B = B \cup A$	5'-	$A \cap B = B \cap A$	(commutative)
6-	$A \cup (B \cup C) =$ $(A \cup B) \cup C$	6'-	$A \cap (B \cap C) =$ $(A \cap B) \cap C$	(associative)
7-	$A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$	7'-	$A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$	(distributive)

8- $\emptyset \cap A = \emptyset$ 8'- $U \cup A = U$

9- $\emptyset \cup A = A$ 9'- $U \cap A = A$

10- $A \cup \bar{A} = U$ 10'- $A \cap \bar{A} = \emptyset$ (complementarity)

11- $\overline{A \cup B} = \bar{A} \cap \bar{B}$ 11'- $\overline{A \cap B} = \bar{A} \cup \bar{B}$ (dualization)

12- $\bar{\bar{A}} = A$ (involution)

13- Each of the following relations implies the other two:

$$A \subset B, \quad A \cap B = A, \quad A \cup B = B.$$

There are several points to be made. First, you may wonder why we choose to present just these particular theorems rather than some others. First of all, there are others, e.g., $\bar{U} = \emptyset$, if $A \subset B$, then $\bar{B} \subset \bar{A}$, etc. The reason is simply this: once these theorems have been proved by using the basic definitions and arguing in terms of elements, then you need never do that again. Any other true relation among subsets and these operations can be proved directly from these theorems without recourse to the basic definitions. Here is how it is done for the two examples we mentioned.

Theorem $\bar{U} = \emptyset$.

Proof. By theorem 10', $U \cap \bar{U} = \emptyset$. But by theorem 9', $U \cap \bar{U} = \bar{U}$. Equating these yields the result.

Theorem If $A \subset B$, then $\bar{B} \subset \bar{A}$.

Proof. By theorem 13, $A \subset B$ implies $A \cap B = A$. Taking complements and using theorem 11', $\bar{A} = \overline{A \cap B} = \bar{A} \cup \bar{B}$. From theorem 5, $\bar{A} = \bar{A} \cup \bar{B} = \bar{B} \cup \bar{A}$, and so theorem 13 implies $\bar{B} \subset \bar{A}$.

Thus, if you learn these theorems, it is possible for you to prove any true relationship involving subsets, union, intersection, complementation, and inclusion simply by manipulating them. It must, however, be emphasized that this is by no means the only set of theorems from which the remainder can be derived; there are many such, but this seems to be a very useful one.

Second, we placed a number of these theorems in parallel. Why? If you will examine them carefully, you will see that each pair is in a sense dual. Specifically, take any one of them and make the following changes: replace each set by its complement (keeping in mind that it can be proved that $\emptyset$ and U are complements of each other), interchange union and intersection symbols, and reverse the direction of any inclusion relations. This will yield, in essence, the other theorem on the same line. For example, consider theorem 8, $\emptyset \cap A = \emptyset$. Making the substitutions yields $U \cup \overline{A} = U$, which is almost, but not quite, theorem 8'. In stating theorem 8', and many of the others, we have dropped the complementation sign on the A's and B's which would have made the pairs perfectly dual. The reason for this is that the sets are arbitrary, and we can always insert $\overline{A}$ for A and have a true theorem. Thus, to prove 8' in the form stated, we should have begun with $\overline{A}$ in theorem 8, Aside from that, there is a perfect duality.

Third, let us comment on each of the theorems so that you will gain a fuller understanding of their meaning.

1 and 2 have already been discussed when inclusion was first presented. (Section 1.3).

3-$\emptyset \subset A \subset U$ simply states the fact that the empty set is a subset of every set and that every set under discussion is a subset of the universal set.

4-The idempotent laws, $A \cup A = A$ and $A \cap A = A$, are quite different from anything you are familiar with from arithmetic. If you "add" a set to itself, you do not gain anything; if you ask what is common between a set and itself, you find it is just the set.

5-The commutative laws say that the order of forming union and intersections does not matter. This is just like ordinary arithmetic, both for addition and multiplication. This rule seems so familiar that we often take it too much for granted. There are important mathematical systems -- important in applications -- where it does not hold. Matrix multiplication is a case in point. Or if you devise an algebra for the operations in a machine shop, which is possible, not all operations will commute: to drill and then thread is hardly the same as to thread and then drill.

6-The associative laws, $A \cup (B \quad C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$, state that it does not matter how you form unions of more than two sets or how you form intersections of them. It does not apply to mixtures of unions and intersections! It is necessary to have such a law, for both operations were defined only for pairs of sets. This rule is just like those in arithmetic, and here as there one customarily drops the parentheses and simply writes $A \cup B \cup C$, $A \cap B \cap C$, etc. But don't drop parentheses when there are mixtures of the union and intersection symbols. The symbol $A \cup B \cap C$ is ambigious. Does it mean $A \cup (B \cap C)$ or $(A \cup B) \cap C$? Draw a Venn diagram to see how different these are.

7-The distributitive laws tell you how to expand (or equally, to contract) mixtures of unions and intersections. If you think of union as analogous to sum in arithmetic and intersection as analagous to product, then the analogue of the first theorem is

$$x\,(y + z) = xy + xz,$$

which is a property of numbers. The analogue of theorem 7', however, is

$$x + (yx) = (x + y)\,(x + z),$$

which, of course, is false. Thus, while there are some parallels to ordinary arithmetic, they are by no means perfect.

8 and 9-These are fairly straightforward: The null set has nothing in common with any other set; any set adjoined to the universal set still yields the universal set; the null set adjoined to any set does not alter it; and any set has exactly itself in common with the universal set.

10-This property of complements is obvious from the way they are defined: a set and its complement have nothing in common and together they exhaust the universal set, i.e., they form a partition of the universal set.

11-We proved one half of theorem 11; but, as they are both very important, you should explore the other with Venn diagrams. In words, these theorems say that the complement of a union is the intersection of the complements, and the complement of an intersection is the union of the complements.

12-The complement of the complement of a set leaves the set unchanged. In effect, this is a case of double negation leaving things unchanged.

13-This theorem says, in essence, that we have defined too many concepts as primitive, that we have been redundant. We could, for example, have only defined union and equality, and in terms of these introduced intersection and inclusion. Other combinations are possible, too. We did not do this because each of the concepts is in one way or another so important and because the duality is so much clearer if they are all involved. It is, nonetheless, well to keep theorem 13 in mind, for it is often convenient to translate an inclusion relation into one of the other two forms. We did this when proving $A \subset B$ implies $\overline{B} \subset \overline{A}$ from the theorems.

Most often these theorems are used to simplify expressions which arise in problems involving sets, just as in ordinary algebra you use the basic rules to simplify an expression of the form $(x + y) z + (x - z) (x + y)$ to $x (x + y)$. Two examples will illustrate this. Suppose the set $\overline{(A \cap \bar{B})} \cup B$ is given.

$$\begin{array}{rll} & & \text{using theorem} \\ \overline{(A \cap \bar{B})} \cup B & = (\bar{A} \cup \bar{\bar{B}}) \cup B & 11' \\ & = (\bar{A} \cup B) \cup B & 12 \\ & = \bar{A} \cup (B \cup B) & 6 \\ & = \bar{A} \cup B. & 4 \end{array}$$

Similarly,

$$\begin{array}{rll} (A \cup \bar{B}) \cap (\bar{A} \cup B) & = [(A \cup \bar{B}) \cap \bar{A}] \cup [(A \cup \bar{B}) \cap B] & 7 \\ & = [\bar{A} \cap (A \cup \bar{B})] \cup [B \cap (A \cup \bar{B})] & 5' \\ & = [\bar{A} \cap A) \cup (\bar{A} \cap \bar{B})] & 7 \\ & \quad \cup [(B \cap A) \cup (B \cap \bar{B})] & \\ & = [\phi \cup (\bar{A} \cap \bar{B})] \cup [(B \cap A) \cup \phi] & 10' \\ & = (\bar{A} \cap \bar{B}) \cup (B \cap A). & 9 \end{array}$$

Problems

1-In a Venn diagram of sets A, B, and C, identify the following sets

$(\bar{A} \cup B) \cap C$; $(A \cup B) \cap (\bar{A} \cap B)$; $(A \cap C) \cup (A \cap \bar{C})$.

2-Represent the two expressions $(A - B) - C$ and $A - (B - C)$ in terms of A, B, C, and the operations $\bar{\ }$, $\cup$, and $\cap$. Simplify as much as possible.

3-Simplify the following expressions:

$A \cap (A \cup B)$; $A \cup (\bar{A} \cap B)$; $(\overline{A \cup B}) \cap \bar{B}$; $(A \cap B) \cup (A \cap \bar{B})$.

4-Using the basic definitions of union, intersection, complement, and inclusion, prove theorems 7 and 11'.

5-Prove each of the following relations in two ways (by using the basic definitions without recourse to theorems 1-13, and by using theorems 1-13 without ever considering an element):

$A \cap B \subset A$ and $(A - B) \cup B = A \cup B$.

7

1.7 SET FUNCTIONS

This section is really an aside here; logically it should not come until near the end of the next chapter. But because the idea of a set function enters near the beginning of the probability course, it seems advisable to bring it in early here. In essence, all we want to say now is that in various ways sets can have numbers attached to them, and often such numbers are of interest. For example, "attached" to any finite set A is the number of its elements, which is usually denoted by $| A |$. For example, $| \{1,5,11,65\} | = 4$. In addition, depending upon what A is and what our purposes are, there may be other numbers. If A is a class of students, their average grade is a number associated with A. Note that it is actually associated with A as a whole, since it is an average grade, and not to any of its elements - the students. If A is a set of banks, there is a highest interest rade among the banks in A, and this is a number attached to A. Indeed, all sorts of aggregated measures which do not apply to individuals (people, banks, industries, countries, or what have you) but do apply to sets of individuals are examples of what we mean.

Let us return for the moment to the number $| A |$, the number of elements in a finite set. Suppose U is a finite universal set, then for each subset A, the number $| A |$ is defined. In a way, this yields something like a function in the calculus or in algebra, something like x^n or log x or sin x. With these more familiar functions one has an independent variable x, which ranges over the real numbers, and to each value of x another number is assigned. In analogy, we take a generic subset A of U as the independent variable, which ranges over all the subsets of U, i.e., over 2^U, and to each value of the independent variable - to each subset - a number is assigned, $| A |$. This may seem to be stretching the usual terminology for functions pretty far, especially since we don't seem to have any formulas to work with as in algebra or trigonometry. We'll come back to this question of formulas later, for things are not quite as they seem. For now, it will suffice to say that a number of useful things can be done with such an extended idea of a function.

In summary, then, if U is any set, a *real-valued set function* is any assignment of numbers to the subsets of U.

Probably the most widely used set functions are those arising in the theory of probability. Although we shall not delve into this here, a simple example will suggest how they arise. If a die is thrown only once, there are six possible outcomes: either a 1, or a 2, or.... or a 6 will come up. Thus, we may take $U = \{1,2,3,4,5,6\}$ as the set of primitive events, one of which will occur. Suppose we assume that the die is perfectly balanced, so the probability of each of these events occurring is 1/6. Now, we can also consider somewhat more complicated events, for instance let us say that the event A has occurred if the die comes up either 1,3,4 or 6, and not if either 2 or 5 appears. We see that A is a subset of U, namely $\{1,3,4,6\}$. The event A has a certain probability of occurring which can be computed from the basic probabilities; it is, of course, 4/6. In a similar way we can assign probabilities to each of the possible complex events, i.e., to each of the subsets U. This set function, and others like it, are known as *probability measures*. Of course, in more general contexts the events are far more complex and the assignments of probabilities are not so simple, but this example illustrates the general case reasonably well.

By and large, the functions of interest in algebra and trigonometry have very restrictive properties. For example, a central property of the logarithm is that

$$\log (xy) = \log x + \log y.$$

Most of the functions which arise from theoretical considerations in physics are not highly arbitrary assignments of one variable to another, but are strongly constrained in one way or another. In fact, these constraints are vital to much of the computational power characteristic of physics. It is similarly true that when the concept of a function is generalized to set functions, or to even more general functions, we still will be most interested in those which possess various sorts of inner constraints. Roughly, the value of the function at one argument will be closely related to its values at other arguments which are themselves related to the first argument, as, for example, x, y, and xy were related above. Consider the function: the number of elements in a subset. It can be shown that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

When a set function arises, one must always be on the outlook for such relationships, for they can be most important.

This whole topic will be resumed and discussed more fully in Chapter 3.

Problems

1-Prove $|A \cup B| = |A| + |B| - |A \cap B|$.

2-Show $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

3-Give an example of a non-trivial industrial set function.

4-Does $|A| = |B|$ imply $A = B$? Does $A = B$ imply $|A| = |B|$?

1.8 ALGEBRAS OF SETS

So far we have tacitly assumed that once a universal set is given, we will be interested in all of its subsets. This, however, is not always the case in practice -- certain subsets may for one reason or another be distinguished as important, others not. Consider U to be the employees of a company. Most of the possible subsets would have little or no functional meaning in the operation of the company, and so will not receive any attention as wholes. But others will be treated as whole units for at least some purposes. For example, the subsets corresponding to departments in the company may be important. If so, then it usually follows that the people (if there are any) common to two departments are also important -- they may have more power or information than the other members of their departments. Also the subset which consists of the union of two departments presumably is an important unit. For example, they may form an operating coalition against the rest of the company. Finally, if one isolates a department as important, then its corporate environment -- all the rest of the company -- also bears consideration. In other words, if we single out a class of subsets as important, it is more than reasonable for us to include their unions, intersections, and complements as also important. But why stop at this level? What of the unions, intersections, and complements of these new sets, and so on. Eventually, this process will stop in the sense that any "new" union, intersection, or complement is not really new; it is one of the subsets already included. At first, one may think that this procedure would necessarily generate all the possible subsets of U, but this is by no means necessarily so. For example, if we begin only with the subsets ϕ and U, we will never get more than these two sets. In the industrial example, we will only get subsets of people closely related to departmental lines, and not many of the crazy subsets which criss-cross departments without any rhyme or reason.

The important thing to notice is that in filling out one of these classes of subsets, we stop when the following properties are met: the union of any two sets from the class is again in the class, the intersection of any two sets from the class is again in the class, and the complement of any set from the class is also in the class. In other words, the class is closed under the operations of union, intersection, and complementation -- closed in the sense that we cannot get outside it by means of these three operations. This is a very familiar property of many systems we know: the sum of two numbers is again a number, i.e., numbers are closed under addition; the vectorial composition of two forces is again a force; etc. By and large in mathematics we are interested in operations which are closed, for then we are free to perform the operations whenever we like without concerning ourselves whether we will get outside the set of elements we are interested in.

So in summary, we make the following:

Definition: Let $\mathcal{A}$ be a class of subsets of a given set U. $\mathcal{A}$ is an algebra of sets if whenever $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$, and $\overline{A} \in \mathcal{A}$.

As suggested above, $\{\phi, U\}$ is an algebra of sets -- the smallest one. This is easily verified.

Suppose we know that we want a particular non-empty proper subset A of U to be in an algebra of sets, but so far as we are concerned it does not matter if any other subset is in $\mathcal{A}$. Then, we might look for the smallest algebra of sets which contains A. We know that if $A \in \mathcal{A}$, then $\overline{A} \in \mathcal{A}$, so we have to have at least $\{A, \overline{A}, \phi, U\}$. The question is: do we have to add on anything more? No, as you can easily check by applying the definition: $\{A, \overline{A}, \phi, U\}$ is, indeed, an algebra of sets.

Of course, the set of all subsets of U is also an algebra of sets -- the largest possible for U -- since any set operation on subsets of U yields a subset of U.

Given any arbitrary class of subsets of U, it is always possible to find the smallest (it is unique!) algebra of sets containing the given class.

Problems

1-Let $U = \{a, b, c, d\}$. Construct the smallest algebra of sets containing $\{a\}$ and $\{a, b\}$. Construct the smallest one containing $\{a\}$ and $\{b, c\}$. Compare these two.

2-Let $A, B \subset U$. Show that the smallest algebra of sets containing A and B is the same as the smallest one containing A, $A \cup B$, and $A \cap B$.

3-Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras of subsets of U. If $\mathcal{A} \subset \mathcal{B}$, show that for each $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $A \subset B$.

1.9 LEGISLATIVE SCHEMES

Legislative bodies and committees that reach decisions by voting according to fixed legislative schemes are an ever present part of modern western life. For the most part, they tend to be organized along traditional tested lines, such as simple majority rule where each person has a single vote. But in some cases, new quirks are suggested and sometimes adopted -- an important case in recent history being the Security Council of the United Nations, where a crude attempt was made to reflect the differential power among the nations. Even the now traditional congressional system of the United States included some variations which had not been tried when it was first adopted. There is often some ambiguity about what a new set of verbal rules implies for the actual operation of a legislature, and the problem is whether we can devise a systematic way to see through the verbal formulation of a voting scheme to its actual implications. In this section, we propose to use the tools so far introduced to lay the ground work for one such analysis, and then in Chapter 3, when we have still more tools, we will present and criticize that analysis.

A legislative scheme, in contrast to a legislature which refers to the specific people with all their peculiarities and affiliations working within a scheme, is a system of rules which state the conditions under which a bill (or motion or what have you) is passed. Ignoring some special classes of bills (treaties, for example) and the problem of ties, the rule which characterizes the United States legislative scheme is this: a bill is passed either if a simple majority in each house of Congress and the President votes for it, or if a two-thirds majority in each house of Congress votes for it. The scheme is given abstractly without any reference to who is in the legislature, what forces are acting upon them, what party structure there is, etc. To be sure, all of these and more are important facts to know when trying to understand the behavior of a particular legislature, but they are quite irrelevant in evaluating or understanding the scheme itself.

As a result, we shall want to deal with legislative roles, not legislators. Thus, we suppose U is the set of legislative roles. Then, the legislative scheme singles out certain subsets of U as able to pass a bill. We shall call these subsets winning coalitions -- coalitions to emphasize the cooperative nature of the process. All other conceivable coalitions are unable to pass a bill and so they are called losing. Thus, a legislative scheme, which is usually given in the form of verbal rules, is equivalent to listing the winning coalitions, i.e., to giving a subset W of 2^U.

A few seconds thought will indicate that not just any old subset of 2^U will do as a possible legislative scheme. There are certain characteristics common to all legislative schemes which serve to put constraints on the possible subsets of 2^U. First, there is always at least one way that a bill can be passed, so W must not be the empty set. Second, it would never do to have both a set A and its complement, $\bar{A}$, both winning coalitions, for then both the bill and its negative could be passed. All known schemes avoid this possibility. Third, the addition of more votes to an already winning coalition always results in a winning coalition. There is no logical necessity for this condition, and not having it would surely make legislative bargaining a more exciting and subtle activity than at present, but it always seems to be met and it has a certain compelling ethical quality.

These we shall take as the conditions characterizing a legislative scheme. But, you may protest, there are a number of other conditions which seem just as basic -- at least, they are found in all legislative schemes. For example, the empty set is always losing, a subset of a losing coalition is always losing, two disjoint coalitions cannot both be winning, etc. These, as we shall see, follow logically from the ones we have singled out. Furthermore, it is our contention that any new condition you propose is either a logical consequence of the ones we shall assume or we can find an example of a legislative scheme where it does not hold.

Let us summarize all of this compactly as a definition.

Definition: Let U be a given finite set, $W \subset 2^U$, and $L = 2^U - W$. We shall say that W is a legislative scheme if these conditions are met:

i. $W \neq \emptyset$,
ii. if $A \in W$, then $\bar{A} \in L$,
iii. if $A \in W$ and $A \subset B$, then $B \in W$.

While this is not a very rich mathematical structure, still it is possible to prove a few trivial theorems of the sort mentioned above. We shall list five and prove the first three; the last two are presented as problems.

Theorem 1- $U \in W$

Proof. Since by i, $W \neq \emptyset$, there exists some $A \in W$. But $A \subset U$, so by iii, $U \in W$.

Theorem 2- $\emptyset \in L$.

Proof. By theorem 1, $U \in W$, so by ii, $\emptyset = \bar{U} \in L$.

Theorem 3- If $A \in L$ and $B \subset A$, then $B \in L$.

Proof. Suppose, on the contrary, $B \in W$, then since $B \subset A$, iii implies $A \in W$. This is contrary to the hypothesis that $A \in L$, so $B \in L$.

Theorem 4- If $A, B \in W$, then $A \cap B \neq \emptyset$.

Although we have demanded what the complement of a winning coalition be losing, we certainly have not made the assumption that the complement of a losing coalition must be winning. In many schemes, such as simple majority rule with an odd number of participants, it is true; in others, however, it is not. In the Security Council it is quite possible to have two factions each of which is able to block the passage of a motion. For this reason, the coalitions in the set

$$B = [\ A \mid A \in L \text{ and } \bar{A} \in L\ ]$$

are called blocking coalitions.

Theorem 5- $B \neq L$.

The main purpose of formulating legislative schemes mathematically is certainly not to prove such theorems as these -- they are much too trivial to be of any interest in and of themselves. Rather, we want to lay out in abstract form what one can mean by such a scheme so as better to be able to see the implications of a particular scheme and to compare several competing schemes. Once it is seen as a mathematical system, then one can use mathematical techniques and reasoning to get at the implications involved. As an example, consider the following schemes:

1-a three-man committee in which each person has one vote and the the decisions are reached according to majority rule; and

2-a three-man committee in which man 1 has two votes, and each of the other two men have a single vote. Decisions are made according to majority rule, except that when there are ties man 2 breaks the deadlock. (Such a scheme might arise if each of the men represented a faction, and it was deemed that the first faction was stronger than either of the other two, and the second was somewhat stronger than the third.)

It is perfectly evident that the first scheme is egalitarian, giving each member of the committee equal weight. The second scheme is equally clearly not egalitarian: the third man is in a far weaker position than the other two. It is not quite clear intuitively how much better off man 1 is than man 2, for although 1 has two votes, 2 can break ties. These differences are intuitively clear and the way to show it conclusively is, of course, to look at the winning coalitions. In scheme 1 they obviously are

$$W = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

In scheme 2, we must look more carefully. There is here the possibility that the one man coalition $\{1\}$ may be winning. However, against $\{2, 3\}$ that would result in a tie, which 2 breaks, so $\{2, 3\}$ is winning and $\{1\}$ is not. It is easy to see that $\{1, 2\}$ and $\{1, 3\}$ are also winning, and so, of course, is $\{1, 2, 3\}$. But this is the same set of winning coalitions, so in point of fact these two apparently different sets of rules are identical.

If you look back at the rules, you will immediately see through them, but the point is that you probably didn't at first. As the size of the committee increases and as the rules are made more complex, it is less and less likely that one will be able to see their implications unless he carries out some sort of formal analysis. Simply listing the winning coalitions is one way. This can be tedious however. In Chapter 3 we will describe a general formal analysis which is applicable to the a priori evaluation of power in a legislative scheme and in a wide variety of other somewhat related situations.

The remainder of this section is devoted to laying the background for these pursuits and as an illustration of the use of set functions.

Suppose we attach the number 1 to each of the winning coalitions in a legislative scheme and the number 0 to the losing coalitions, then we have a set function which is simply equivalent to stating the winning coalitions. Formally, such a function would be introduced as follows:

$$v\ (A) = \begin{cases} 1, \text{ if } A \in W \\ 0, \text{ if } A \in L \end{cases} .$$

This function v will be known as the characteristic function of the legislative scheme W. Since we imposed some restrictive conditions on W, it must follow that v also meets some restrictive conditions. We could state a whole variety of them, but they would not all be independent of each other. So we shall choose a particular set of three which are fairly standard in literature, namely:

i. $v\ (\emptyset) = 0$,
ii. $v\ (U) = 1$,
iii. if A and B are disjoint,
$v\ (A \cup B) \geq v\ (A) + v\ (B)$.

These we prove: The first follows from theorem 2 above. The second is an immediate consequence of theorem 1. The third is slightly more complicated. If A and B are both losing, the right side $v\ (A) + v\ (B) = 0$ so the inequality or the equality holds. If they are not both losing, then by theorem 4 only one is winning, and by condition iii on W, $A \cup B$ is winning. So in that case, $v\ (A \cup B) = 1 = v\ (A) + v\ (B)$; hence the equality holds.

We claim that no other conditions independent of these follow from the assumptions about W. To show this, we will prove that any set function having values only 0 and 1 and meeting these conditions defines a legislative scheme if we take $W = [A \mid v\ (A) = 1]$.

i. $W \neq \emptyset$, since $v(U) = 1$ implies $U \in W$.

ii. If $A \in W$ and $\bar{A} \in W$, then $v(A) = 1 = v(\bar{A})$, and so

$$1 = v(U) = v(A \cup \bar{A}) \geq v(A) + v(\bar{A}) = 1 + 1 = 2,$$

but this is impossible. Thus, we must conclude that A or $\bar{A} \in L$.

iii. Suppose $A \in W$ and $A \subset B$. From the properties of sets it is easy to show that $B = A \cup (B - A)$ and that $A \cap (B - A) = \emptyset$, so

$$v(B) = v[A \cup (B - A)] \geq v(A) + v(B - A) \geq 1.$$

but since v assumes only the values 0 and 1, this means $v(B) = 1$, hence $B \in W$.

Thus, the idea of a legislative scheme and of a set function with values 0 and 1 and meeting these three conditions are essentially the same. One virtue in noting this identity is that we have transformed our qualitative problem into one involving numbers, and so we may be able to use some of the quantitative mathematics about which so much is known.

It also turns out that we have arrived at some conditions on this set function which are extremely important in a part of game theory. From time to time we shall mention parts of game theory as illustrative of an application of some of our mathematical ideas, and, in fact, before we are done we will have sketched some of its central features. The central problem studied in game theory is this: several people -- called players -- are in a situation where each has a number of possible courses of action. Depending upon which courses are elected by the several players, there will be different consequences for them. Each player is supposed to have preferences (not in general the same pattern of preferences for all the players) among these consequences, and he is assumed to try to select his action so as to get what he wants. The complication for him, and for the theory builder, is that his outcome depends not only upon what choice he makes, but upon the choices of each of the others. The only information he has about the other players is their preference patterns and that they too are trying to choose their action so as to get what they want. The problem is to use this information to guide action and to predict what will happen.

At present the theory divides into several parts. One important distinction is whether there are only two or more than two players. And when there are more than two, another distinction is whether they are free to cooperate with one another if they wish or not. In the case where cooperation is permissable, one proceeds as follows: Suppose that a coalition A -- a subset A of the set of all players -- elects to form and to cooperate, then it will be faced by some sort of opposition from the other players. The worst possible case it can meet is if the remaining players, $\bar{A}$, also form a coalition; this is the worst because $\bar{A}$ can do everything, and possibly more, than any less unified opposition. So a conservative evaluation of A's strength is obtained by examining the two "person" game which results when A and $\bar{A}$ are pitted against one another. Using the theory of two person games, which we shall not go into here, it is possible to obtain a suitable numerical measure of this strength; this number we denote by $v(A)$. If this number is computed for each possible coalition, then a real valued set function v results. It is known as the characteristic function of the game.

It is not by coincidence that we have used the same symbol and name for this function as for the one introduced in connection with a legislative scheme, for it can be shown mathematically that the characteristic function of any game must meet two, and only two conditions:

i. $v(\emptyset) = 0$,
ii. if A and B are disjoint subsets of U,
$$v(A \cup B) \geq v(A) + v(B).$$

These conditions say, in effect, that the null set has no strength, and that the union of two disjoint coalitions is never weaker than the sum of the strengths of these two coalitions taken separately. The union can do everything the separate coalitions can, and possibly more.

It is not unreasonable that a measure of the strategic possibilities in a legislative scheme should be a special case of such a measure for games in general, since voting on bills is a conflict of interest problem which ought to be encompassed in some fashion by the theory of cooperative games.

We will not try to push these ideas any further now, for we are in need of more tools. In Chapter 3 we will continue our program of evaluating the power structure of legislative schemes; however, there is no reason to restrict it to that special case, so we shall cast it in the framework of general characteristic functions.

Problems

1-Prove theorem 4: if A, B $\in$ W, then A $\cap$ B $\neq \phi$.

2-Prove theorem 5: B $\neq$ L.

3-Examine the following legislative schemes by presenting the sets W, L, and B:

a-a four man committee (a,b,c,d) in which they have 2, 1, 1, and 2 votes, respectively, under majority rule, and where man b can can break ties. Note this is case 2 above of a three man committee with a two vote fourth man added. What has happened to man 3?

b-(a,b,c,d) in which they have 4, 3, 2, 1 votes, respectively, under majority rule, and where the chairman can break ties. Show that securing the chairmanship is equivalent to obtaining an additional vote.

c-(a,b,c,d,e) in which they have 1, 1, 1, 3, and 5 votes, respectively, under majority rule; man a has veto power which can be overridden by a 2/3 majority.

CHAPTER II

RELATIONS, ORDERINGS, AND FUNCTIONS

2.1 PRODUCT SETS

Having explored something of the generation of new sets from old by selecting smaller sets -- subsets -- from a given set, we turn in this chapter to questions of building up larger sets from two or more smaller ones. An automobile manufacturer may advertise that his cars come in ten colors and six models, giving the customer a choice from among 60 combinations. It is obviously much more compact to list the set of ten colors and the set of six models separately than to list all 60 combinations. Similarly, a menu listing ten appetizers, three soups, twenty entrees, five vegetables, ten desserts, and four beverages offers the diner a choice from among 120,000 complete menu ~~means~~. Only the paper industry could want this set listed in explicit detail. Everywhere you look you will find enormous sets presented compactly as several much smaller sets with the indication that the overall set is generated by making a single choice from each of the simpler sets. Each of these is an example of what is known in mathematics as the product of several sets.

The easiest case to deal with is only two sets A and B. Then the set of elements of the form (a,b), where $a \in A$ and $b \in B$, is known as the (Cartesian) product of A and B; it is denoted by the symbol $A \times B$. In order to encompass the menu and many other examples, we must define this concept for more than two sets. For that example we have sets A (standing for appetizers), S (for soups), E, V, D, and B, and the set of all possible meals, $A \times S \times E \times V \times D \times B$, consists of all possible elements of the form (a,s,e,v,d,b), where $a \in A$, $s \in S$, etc. In general when you have more than two sets it is kind of messy and, if there are enough of them, taxing to use totally different symbols for each. So it is customary to use a single generic symbol for all the sets and to differentiate among them by indices. If there are n sets, it is simplest to index them $A_1, A_2, \ldots, A_n$. Using this notation, then we can make the following general

Definition: Let the sets $A_1, A_2, \ldots, A_n$ be given. The (Cartesian) product of these n sets is defined as

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

The symbol on the left is often abbreviated by $\prod_{i=1}^{n} A_i$.

It is important to realize that a definite order is involved in the several entries of the elements of the product space. The first entry is always filled by an element from A_1, and not from any of the other sets, the second from A_2, and so on. This is not to say that the same element might not be in two or more of the sets, or indeed that several of the sets may not be identical, but rather that they are distinguished as playing different roles by their ordering in the product set. The fruit cup in the set of appetizers is distinguished from the fruit cup in the set of desserts only by being in the one set rather than the other.

In engineering and physical problems one product space is almost painfully familiar, namely the coordinate system drawn in Fig. 11. To see that this is a product space, let X denote the set of points on the x axis and Y the set of points on on the y axis, then the set of points in the whole plane specified by these axes is simply $X \times Y$. If we want to work with a three dimensional Euclidean space, then we add the third coordinate Z and the whole space is $X \times Y \times Z$. Clearly, this can be generalized to n dimensions. Since in physics it is often useful to think of the point (x,y) in the plane as specifying a vector from the origin to that point, it is customary to call the elements of

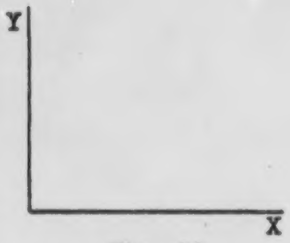


Fig. 11

$X_1 \times X_2 \times \ldots \times X_n$, where each of the X_i are the sets of real numbers, (n-dimensional) vectors. The term n-tuple is also widely used.

If $S = A_1 \times A_2 \times \ldots \times A_n$, then we speak of each A_i as a component of the product set S. Components play much the same role as the coordinates of a geometrical space, though, of course, they need not be the number system or any other particular set. Since we know that the coordinates of a geometrical space are not unique -- any rigid rotation will do just as well -- we cannot in general expect a product space to have a unique decomposition into components.

Problems

1-Present a non-trivial industrial example of a product space.

2-Let A = {male, female}, B = {old, young}, and C ={skilled, unskilled}. Write out all of the elements in $A \times B$ and in $A \times B \times C$.

3-Using the same sets as in problem 2, express the following set (which, in words, consists of all categories of old workers) in the most compact way that you can: {(male, old, skilled), (male, old, unskilled), (female, old, skilled), (female, old, unskilled)}.

4-Suppose $S = A \times B$ and $T = C \times D$. What does it mean if someone asserts S = T ?

5-Suppose A = {0,1,2,3,4,5,6,7,8,9}. Where have you seen $A \times A$ arise?

2.2 RELATIONS

The familiar term "relationship" connotes a whole class of properties which relate one individual to another individual of the same general type -- those properties X which appear (at least implicitly) in sentences of the form "a has the relationship X to b." For example, Mr. Smith is a superior of Mr. Jones. Here the relationship is "is a superior of" and it holds among people. Other common examples of relationships are "likes" between people "is in a state of war with" between countries, "is less than" between numbers, "is the mother of" between people, etc.

The crucial features about these examples seem to be three: First, a relationship holds between pairs of things of the same general type. Second, it generally holds only between some pairs, and not between others. For example, "mother of" holds only between certain selected pairs, namely: each mother and her daughters and sons. Third, sometimes the order in which the two elements are taken matters: if Jane is the mother of Mary, then Mary is not the mother of Jane. In other cases, the order may not matter, such as "in a state of war."

Suppose a set A is given and that R denotes a relationship, i.e., a property which may or may not hold between ordered pairs of elements. If $a,b \in A$, let us write aRb if the relationship holds from a to b. If it does not hold, we write $a\bar{R}b$. The list of all pairs (a,b) such that aRb is said to be the relation on A induced by the relationship R. Actually, we will use the same symbol R to denote both the relation on A and the relationship which induces it; there is a slight ambiguity here, but it is not really serious.

But what is a listing of all these pairs (a,b)? Simply a subset of $A \times A$. It is the subset which is singled out by the given property R. Conversely, given any subset of $A \times A$ one can always find a relationship which singles it out. Thus, we are led to the following

Definition: An (abstract binary) relation over a given set A is a subset of $A \times A$. If R denotes the relation, i.e., subset of $A \times A$, we write aRb if $(a,b) \in R$ and $a\bar{R}b$ if $(a,b) \notin R$.

The prefix "binary" to the word relation is needed because we choose to deal only with pairs of elements; there are, of course, trinary relations (subsets of $A \times A \times A$), etc., but these seem to be of considerably less importance. The word "abstract" is also prefixed because we have not specified the property which singles out the subset R of

$A \times A$. For a given set A, it can easily happen that two quite distinct relationships single out the same subset, in which case we have two different realizations of the same abstract relation. For a specific set of people, it is entirely possible for the relationships "is a friend of" and "works with" to be identical, though in general they are distinct.

Problems

1-Let A = {United States, Great Britian, Germany, Japan, Russia}. Write out the relation " was at war in 1944 with" over A; the same thing in 1939.

2-Can you see how to treat a business flow chart and an organizational diagram as a relation? Give a simple example with which you are familiar, explicitly stating what relationship is involved and what the relation is.

2.3 THREE IMPORTANT SPECIAL PROPERTIES

Here, as almost everywhere in mathematics, one continually has his eye out for the recurrence of the same general property in a number of important relations. If such a property is detected, it is often useful to isolate it and to see how it interlocks with others you already know about. We have done a little of this sort of thing before, e.g., when we isolated those classes of subsets called algebras of sets. And we will continue to do it. In this section we shall be concerned with three general requirements on relations which have loomed very important in mathematics.

First, we shall consider relations in which aRa holds for every $a \in A$. This is true for the relation "less than or equal to" between numbers. It holds by virtue of the fact $a = a$, so, trivially, $a \geq a$. It is true for the relationship "lives in the same house as," since a person certainly does live in the same house as he does. It is so tautological in these cases, one might begin to wonder if it doesn't always hold, but "mother of" quickly dispells that conjecture. In that relationship, not only does aRa fail for at least one $a \in A$, but $a\bar{R}a$ for every $a \in A$. The same is true for "greater than" between numbers. Still other relations have aRa for some $a \in A$, and $a\bar{R}a$ for the remainder. An example is the relationship "depreciates," for some people depreciate themselves, others do not.

> *Definition*: Let R be a relation on the set A. R is said to be *reflexive* if aRa for all $a \in A$; it is said to be *irreflexive* if $a\bar{R}a$ for all $a \in A$; and it is said to be *non-reflexive* otherwise.

In these terms, "less than or equal to" and "lives in the same house as" are reflexive; "mother of" irreflexive; and "depreciates" is in general non-reflexive. It will be recalled that we spoke earlier of inclusion among subsets as being reflexive. This is compatible with the present definition since it is easy to see that inclusion among the subsets of U is a relation on 2^U which is reflexive.

In a good many applications, it is a question of convention, convenience, or taste whether or not to interpret a relation as reflexive. For the relationship "is in a state of war with" one must decide whether to treat a civil war or a revolution as a war between a country and itself. For "communicates to" shall we say a person communicates to himself or not? If we say that "a is the brother of b" when a and b have the same parents, then "brother of" is reflexive, but we would just as easily define it so that it is irreflexive. A certain amount of judgment is sometimes needed in these ambiguous cases.

We turn to the next general category of relations. Earlier we emphasized that generally the order in which we write the elements involved in a relation is material, that aRb is quite a different thing from bRa. Think of "greater than" or "mother of." But for some relationships the order doesn't really matter; there is a perfect symmetry.

"Lives in the same house as" is a case in point: if a lives in the same house as b, then b lives in the same house as a. Other examples are: "equality" between numbers or between sets, "is married to," and "is the same size as." On the other hand, there are relations like "mother of" where if aRb we know definitely that $b\bar{R}a$. Still others are of a mixed quality.

Definition: Let R be a relation on the set A. R is said to be *symmetric* if whenever aRb holds, so does bRa; it is said to be *anti-symmetric* if whenever aRb holds, $b\bar{R}a$; and it is said to be *non-symmetric* otherwise.

The third important property, which we have already run into with inclusion, is typified by any comparative concept such as "larger than": if a is larger than b, and b is larger than c, then we know that a is larger than c. This is true of set inclusion, of "greater than or equal to," of "lives in the same house as," etc. The other extreme would, of course, be a relation where if aRb and bRc, then we would know with certainty that $a\bar{R}c$. For example, if a is the mother of b and b the mother of c, then a is the grandmother of c, and so not the mother of c. In general, "in a state of war with" satisfies the same condition, but there are exceptions, as when Communist China, Nationalist China, and Japan were mutually at war -- at least to all intents -- in the middle forties.

Definition: Let R be a relation on the set A. R is said to be *transitive* if aRb and bRc always imply aRc; it is said to be *intransitive* if aRb and bRc always imply $a\bar{R}b$; and it is said to be *non-transitive* otherwise.

2.4 EQUIVALENCE RELATIONS

Any relation which is simultaneously reflexive, symmetric, and transitive is called an *equivalence relation*. This special word is introduced because these relations appear often and play an important role in many mathematical situations. The equivalence relation "lives in the same house as" illustrates vividly the central feature of any equivalence relation: it divides the population into disjoint subsets, namely, the sets of people who live in the same houses. In other words, it induces a partitioning of the given population. Let us emphasize that this feature is not unique to "lives in the same house as;" it is true of all equivalence relations.

We say than an equivalence relation R partitions the set on which it is defined into equivalence classes, which are characterized as follows: any two elements in the R relation are in the same class, and any two not in the R relation are in different classes. Conversely, any partitioning of a set induces the obvious equivalence relation on the set. Thus, the idea of a partitioning and of an equivalence relation are substantially the same.

The best known example of an equivalence relation is, of course, equality. It is in a sense trivial, however, for the equivalence classes of the equality relation each consist of a single element, whereas, in general, at least some equivalence classes will have more than one element. The idea of an equivalence relation is therefore, a slight, but important, generalization of equality. It says in effect that the elements in the same equivalence class are "equal" to each other with respect to the property inducing the relation, even though they are not identical, as they would have to be for equality. More often than not, we are concerned with equality along one dimension or another, but not strict identity. Often we wish to group things with respect to some parameter and to treat them as all equal in the rest of the analysis. This is what one is doing when one groups people according to income levels, or according to religious affiliation, or profession, etc.

Problems

1-For each of the following relationships state their reflexivity, symmetry, and transitivity properties:

is the brother of, sells to, gives orders to, is the ancestor of, implies, is the son of.

2-Prove formally that the equivalence classes of an equivalence relation form a partitioning.

3-Criticize the following "proof" of this erroneous statement: if a relation R on A is symmetric and transitive, then it is reflexive.

Proof. For any $a \in A$, the fact that the relation is symmetric means aRb implies bRa. But by transitivity, aRb and bRa imply aRa, so the relation is reflexive.

*2.5 MATRIX AND GRAPHICAL REPRESENTATIONS OF RELATIONS

Whenever one has actually to work with real subsets, in contrast to making general theoretical statements as we have been doing, there is a problem of how best to present them. As a special case of subsets, the same problem exists for relations; however, just because of their specialness, some convenient methods exist for relations which are not applicable in general. There are two major methods: a systematic tabular one and a less systematic, but often more revealing, graphical one. Neither of these methods is terribly practical if the underlying set has more than, say, 100 elements.

The tabular scheme is based on the almost trivial observation that a relation on a finite number n of elements amounts to nothing more than a two dimensional table with n rows and n columns. In the entries one mark is placed if the relation holds from the element identified with that row to the element identified with that column; another if it does not hold. More specifically, if we number the elements in A from 1 through n, then we put one mark in the entry of row i and column j if iRj, and another if $i\bar{R}j$. The most widely used scheme is to use 1 if iRj, and 0 otherwise. Example: Let $A = \{1,2,3,4\}$ and

R = {(1,2), (2,1), (2,2), (2,3), (2,4), (4,3)}, then the array is

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

This array, ignoring the row and column lables, forms a 4 by 4 matrix with only the entries 0 and 1.

Other possible entries have been suggested and used -- the choice of convention depends very much upon what one wants to find out and how one is going to do it. Among the other suggestions, two will be mentioned. Enter a 1 in the (i,j) entry if iRj and a -1 if $i\overline{R}j$. Let U be some set (usually having some relation to the problem under investigation). Enter U in the entry if iRj, and ∅ otherwise. The first suggestion results in an ordinary real-valued matrix, just as when 0 and 1 are used; the second, with its entries sets, is a new kind of beast known as a Boolean matrix. We will not look further into either of these representations of a relation.

Returning to the 0,1 representation, suppose that the elements of A are people in some industrial establishment and that the relationship under consideration is "communicates to." In practice, there are serious questions as to what one shall define "communicates to" to mean — but presumably it would be defined in such a manner that the president communicates to his vice presidents and not to a foreman or a janitor. We need not worry about such points here. If we choose any two people $a,b \in A$, we may ask: does a communicate directly to b? If not, is it possible for him to so via some intermediary c? Or by several intermediaries? Obviously, there is no problem to answering the first question; we simply look in row a of the matrix representation and determine whether there is a 1 or a 0 in column b. But to answer the second question, we must simultaneously look for a 1 in row a, column c and in row c, column b, and since we do not care who the intermediary is we must do this for each possible c. And when we go beyond two-step connections, the problem rapidly becomes very messy. What we must find is a systematic way of using matrix operations to answer such questions.

Let us denote by R both the relation and its matrix representation, and let R_{ij} denote the entry, either 0 or 1, in row i and column j of the matrix R. We note that the product $R_{ac}R_{cb}$ is 1 if and only if $R_{ac} = 1$ and $R_{cb} = 1$; otherwise, it is 0. Thus, there is a two-step path from a to b via c if and only if $R_{ac}R_{cb} = 1$. But since we do not care which person c is, there is a two-step path from a to b if and only if the sum

$$R_{a1}R_{1b} + R_{a2}R_{2b} + \dots + R_{an}R_{nb} = \sum_{i=1}^{n} R_{ai}R_{ib}$$

is greater than 0. Furthermore, the value of the sum equals the number of people in A who can serve as intermediaries from a to b.

But to anyone knowing matrix algebra, this sum is very familiar; it represents the entry of row a, column b in the matrix obtained by multiplying R by itself -- in R^2. Thus, simply squaring R gives us at once the number of two-step paths between each ordered pair of elements from A. If R represents direct communication, R^2 the two-step ones, it is plausible to conjecture that R^3 gives the number of three-step ones, and in general R^k gives the number of k-step ones. This conjecture is easily verified.

The main virtue of this observation is that it reduces a fairly complicated counting problem to a very systematic procedure -- matrix multiplication -- which can be carried out by a clerk or by a high speed computer if the matrices are very large.

You may wonder what, if any, are the matrix correlates of reflexivity, symmetry, and transitivity. First, it is easy to see that a relation is reflexive if and only if all the entries in the main diagonal are 1's. The relation is transitive if and only if whenever an entry of R^2 is non-zero, the corresponding entry of R is also non-zero. This we can see as follows: If the (a,c) entry of R^2 is positive, then there exists at least one $b \in A$ such that aRb and bRc. But if R is transitive, this implies aRc, and so $R_{ac} = 1$. The converse is equally easy. The symmetry of a relation is not best seen in terms of matrix multiplication, but in terms of the symmetry of the matrix R. Corresponding to symmetry in a relation is perfect symmetry about the main diagonal, as in the following example

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

It is reasonably evident that these three conditions do not combine into a single simple condition for an equivalence relation (which, it will be recalled, is reflexive, symmetric, and transitive). Nonetheless, there is one fact about the matrix of an equivalence relation which is worth noting. If one numbers the equivalence classes from, say, 1 to s, and then numbers the elements in the first equivalence class successively from 1, those in the second successively from the last number in the first class, and so on, then the 1's in the matrix will appear as non-overlapping squares about the main diagonal. For example,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

represents the equivalence relation with the equivalence classes {1,2,3}, {4}, {5,6}, {7,8}.

For computational purposes, the representation of relations by matrices is generally effective, but for "understanding" the relation they leave a good deal to be desired. Most of us do a lot better with some sort of diagramatic representation, of which flow charts, organicational diagrams, and engineering schematics are typical. The generic mathematical term for such drawings is an oriented (topological linear) graph. Formally, an oriented graph is a collection of points and directed lines connecting them, as in Fig. 12.

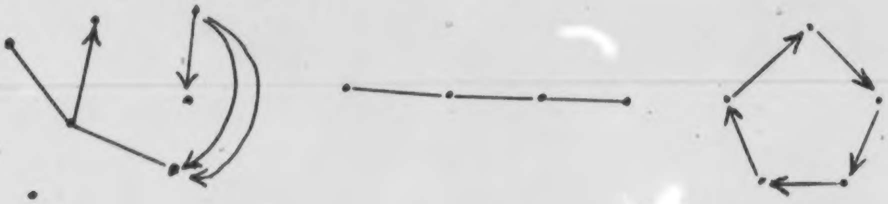


Fig. 12

Let it be clear that this use of the word "graph" is somewhat different from the one with which you are already familiar: the graph of a function on a two dimensional plot.

In diagrams of oriented graphs it is customary to use a single undirected line between points a and b if there is both a directed line from a to b and from b to a. (This we have done in Fig. 12.)

In the general mathematical concept of a graph there may be any number of directed and undirected lines between a pair of points, but we shall restrict our attention to the case where there either is no line at all connecting them, or a single directed line, or a single undirected one. (Terminology: the points are often called nodes or vertices, and the lines, branches or arcs).

It should be clear how to represent a relation by a graph. Distinct points are chosen in the plane, one corresponding to each element of A. If aRb, we draw a directed line from a to b. It does not matter where in the plane we place the points, so long as they are distinct, nor does it matter whether we draw straight or curved lines. All the graphs in Fig. 13 represent the same abstract relation, and they are all equal to one another. (The numbering of the points is introduced to facilitate your seeing the identity of the graphs.) Distances and angles are not at all involved in these representations. On the other hand, there can be great psychological differences among several different graphs of the same relation. Consider those shown in Fig. 14. Most people looking at the first drawing will speak of it as a simple hierarchy. To some people the second suggests "the man behind the throne," and to others a simple hierarchy with a bottleneck. The third elicits the feeling that there is a central person in a focal position of leadership. But, they are the same graph drawn in slightly different ways. This is not to say that vertical organization of the drawing cannot be used to convey some information, but only that it always seems to even when such was not intended.

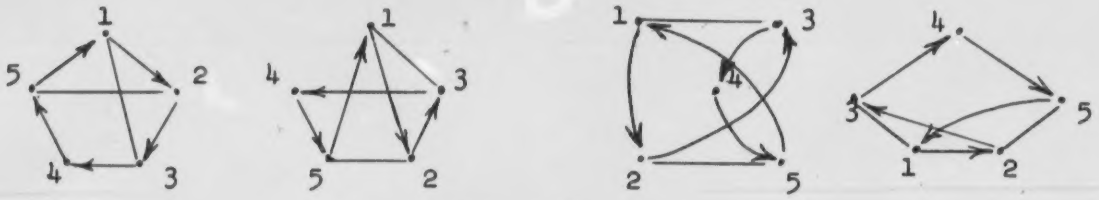


Fig. 13

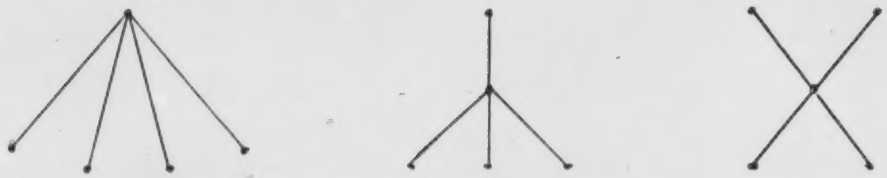


Fig. 14

The graph of a reflexive relation has a closed loop at each point: . The graph of a symmetric relation has only undirected lines. The graph of a transitive relation has no configurations of the type shown at the left of Fig. 15, only those on the right. From these remarks it is

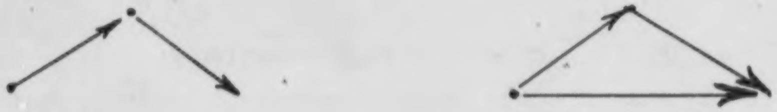


Fig. 15

easy to guess what an equivalence relation must look like: clusters of points with all possible lineswithin each cluster, and none between them. An example (with the closed loops at each note omitted) is shown on the left of Fig. 16. The right hand graph is,

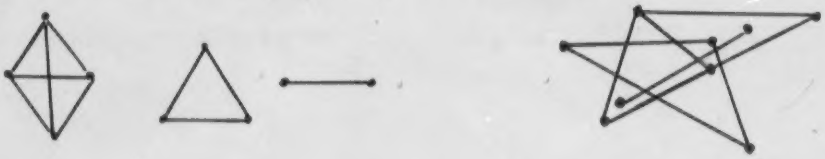


Fig. 16

however, the same relation. We show this to emphasize how difficult it can be to detect the properties of a relation drawn in graphical form if there are no initial hints as to how to organize the drawing. The same remark tends also to hold for matrix representations.

One further graph theoretical idea is needed in the following section. Consider the two unoriented graphs (representations of symmetric relations) shown in Fig. 17. The main difference between these is that the one on

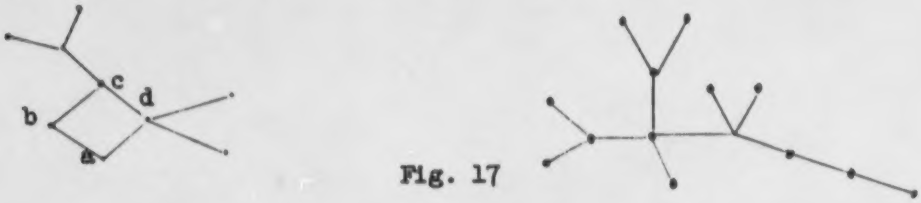


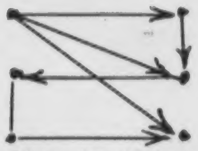
Fig. 17

the left has a closed loop of lines: aRb, bRc, cRd, and dRa; the one on the right does not. Whenever an unoriented graph fails to have such loops, it is called a tree (the reason being fairly obvious). Such graphs play an important role wherever a bifurcating decision process is involved.

Possibly the most extensive application outside of mathematics proper of relations and their representations is in that part of social psychology known as sociometry. The central thesis of this discipline is that certain of the relations which exist and can be observed in groups of people are crucial to an understanding of the behavior of groups, and there is an extensive literature exploring empirical data, relating it to mathematical properties of relations, and probing the mathematics of relations itself. A recent survey of this material is: Lindzey, G. and Borgatta, E. F. "Sociometric Measurement," Handbook of Social Psychology (G. Lindzey, ed.), Addison-Wesley, Cambridge (1954), 405-448.

*Problems

1-Write the matrix representation of the relation having the following graph:



Draw the graph corresponding to this matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

2-For the last matrix, how many three-step paths are there from 1 to 5, from 2 to 4?

3-Can you see any difficulty in the interpretation of the entries of R^k as the k-step paths in the relation represented by R.

4-What does the matrix condition $R^2 = R$ mean for the relation represented by R?

*2.6 GAMES IN EXTENSIVE FORM *

In section 1.9 we mentioned one mathematical construct which arises in the theory of games (which is the current mathematical model for conflict of interest among people or organizations), and here we want to discuss another which illustrates the use of some of the ideas we have so far developed. The material given here arises at the beginning of game theory, when one is first trying to abstract into mathematical form what it is that the rules of a parlor game actually tell you. First of all, the rules of any parlor game specify a series of well defined moves, where each move is a point of decision for a given player from among a set of alternatives. The particular alternative chosen by a player at a given decision point we shall call the choice, whereas the totality of choices available to him at the decision point constitutes the move. A sequence of choices, one following another until the game is terminated, is called a play. Let us suppose that in one game (at some stage of a play) player 1 has to choose among playing a king of hearts, a two of spades, or a jack of diamonds, and that in another game a player, also denoted 1, has to choose among passing, calling, or betting. In each case the decision is among three alternatives, which may be represented by a drawing as in Fig. 18.

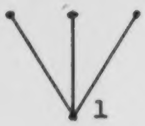


Fig. 18

But how can these two examples be considered equivalent? Certainly it is clear from common experience that one does not deal with every three-choice situation in the same way. One might if they were given out of context, for there would be no other considerations to govern the choice; but in a game there have been all the choices preceding the particular move, and all the potential moves following the one under consideration. That is to say, we cannot truly isolate and abstract

---

* The material in this section is almost identical to p. 39-44 of Luce, R.D. and H. Raiffa, Games and Decisions, John Wiley, New York, 1957.

each move separately, for the significance of each move in the game depends upon some of the other moves. However, if we abstract all the moves of the game in this fashion and indicate which choices lead to which moves, then we shall know the abstract relation of any given move to all other moves which have affected it, or which it may affect.

Such an abstraction leads to a drawing the type shown in Fig. 19 -- to a tree. The number associated with each move indicates which player is to make the move, and therefore these numbers run from 1 through n, if there are n players. In the example of Fig. 19, n = 4, and we see that all the moves, save the first, are assigned to one of the players; the first move has 0 attached to it. A move assigned to "player" 0 is a chance move, as, for example, the shuffling of cards prior to a play of poker. To each chance move, which need not be the first move of the game, there must be associated a probability distribution, or weighting, over the several alternative choices. If a chance move entails the flipping of a fair coin, then there are two choices at the move and each will occur with probability 1/2.

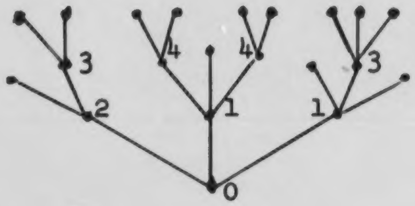


Fig. 19

As we said, the graph of a game is a tree, which is called the game tree. It may not seem reasonable to assume the graph of a game is a tree, for in such games as chess the same arrangement of pieces on the board can be arrived at by several different routes, which appears to mean that closed loops of branches can exist. However, in game theory we choose to consider two moves as different if they have different past histories, even if they have exactly the same possible future moves and outcomes. In games like chess this distinction is not really important and to make it appears arbitrary, but in many ways the whole conceptualization and analysis of games is simplified if it is made. The tree character of a game is not unrelated to the sinking feeling one often has after making a stupid choice in a game, for, in a sense, each choice is irretrievable, and once it is made there are parts of the total game tree which can never again be attained.

The tree is assumed to be finite in the sense that a finite number of nodes, and hence branches, is involved. This is the same as saying that there is some finite integer N such that every possible play of the game terminates in no more than N steps. Such is certainly true of all parlor games, for there is always a "stop" rule, as in chess, to terminate stalemates. To say the tree is finite is not to say that it is small and easy to work with. For example, card games often begin with the shuffling of a deck of 52 cards, and so the first 0 move has 52!, i.e., approximately 8.07×10^{67}, branches stemming from it. Clearly, for such games no one is going to draw the game tree in full detail!

The next step in the formalization of the rules of a game is to indicate what each player can know when he makes a choice at any move. We are not now assuming what sorts of players are postulated in game theory, but only what is the most that they can possibly know without violating the rules of the game. Clearly, there is the possibility that the rules of the game do not provide a player with knowledge on any particular move of all the choices made prior to that move. This is certainly the situation in most card games which begin with a chance move, or where certain cards are chosen by another player and placed face down on the table, or where the cards in one player's hand are not known to the other players. Indeed, it may be that a player at one move does not know, and cannot know, what his domain of choice was at a previous move! The most common example of this is bridge where the two partners must be considered as a single player who intermittently forgets and remembers what alternatives he had available on previous moves.

To suggest a method to characterize the information availa ble to a player, consider a game whose tree is that shown in Fig. 20. The dotted lines enclosing one or more nodes are something new in our scheme; as we

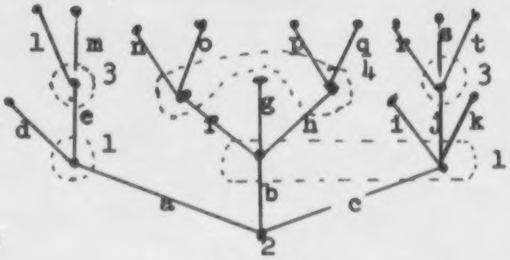


Fig. 20

shall see they can be used to characterize the state of information when a player has a move. Let us suppose that the rules of this game assert that on move 1 player 2 must choose among three alternatives denoted a,b, and c. Regardless of player 2's choice, player 1 has the second move. We shall suppose that the rules of the game permit player 1 to know whether or not player 2 selected choice a. If player 2 chooses b, then the rules are such that player 1 can only know that either b or c was chosen, but not which. While verbally this may seem complicated, graphically all we need do is enclose in a dotted line those moves of player 1 which end up on b or c. The dotted line simply means that from the rules of the game the player is unable to decide where he is among the enclosed moves. The single move at the end of choice a is also enclosed, for if that choice is made player 1 knows it. If choice b was in fact made, and if player 1 then makes choice f (of course, he does not know whether he is making f or i) the next move is up to player 4. Note that according to the diagram, the rules of the game make it impossible for him to determine whether he is choosing between n and o or between p and q.

In general, the rules of any game must specify in advance which moves are indistinguishable to the players -- the sets we have enclosed in dotted lines. Abstractly, there are two obvious necessary features to these sets of moves -- which are known as information sets. Each of the moves in the set must be assigned to the same player, and each of the moves must have exactly the same number of alternatives. For if one move has r alternatives and another s, where r ≠ s, then the player would need only count the number of alternatives he actually has in order to eliminate the possibility of being at one move or the other. A third condition, which may be less obvious, must also be assumed, namely: a single information set shall not contain two different moves of the same play of the game tree. The reason for this is the impossibility of devising rules so that a single player is unable to distinguish between two of his moves which lie on a single play, i.e., on a chain of moves from the first move to an end point of the tree.

Returning to Fig. 20, consider player 1's information set which has two moves. Since they are indistinguishable, each choice on one move must have a corresponding choice on the other move. It is convenient in these diagrams to pair them systematically, so f corresponds to i, g to j, and h to k. It is clear that this correspondence can be generalized to information sets having more than two moves and other than three alternatives at each move.

The final ingredient given by the rules of the game is the outcome which occurs at the end of each play of the game. Almost anything may be found to be the outcome of some game; for example, the subjective reward of victory in a friendly game, or the monetary punishment of seeing someone else sweep in the pot, or death in Russian Roulette. In any given system of rules for a game there is some fixed set of outcomes from which specific ones are selected by each of the plays. Each of the end points of the game tree is a possible termination point of the game and it completely characterized the play of the game which led to that point, for there is only one sequence of choices in a tree leading to a given end point from a fixed first move. We may index these end points and denote a typical one by the symbol x . Now, if X is the set of outcomes, the rules of the game associate to each x an outcome from X which we may denote by $f(x)$. For example, in a game like tic-tac-toe the set of outcomes is {player 1 loses and player 2 wins, player 1 wins and player 2 loses, draw}. In this case, and in a wide class of games, it would be sufficient to state the outcomes for only one of the players, but in other situations which are not strictly competitive it is necessary for the elements of the outcome set to describe what happens to each player.

In summary, then, the rules of any game unambiguously prescribe the following:

i. a finite tree with a distinguished node (the tree describes the relation of each move to all other moves and the distinguished node is the first move of the game);
ii. a partition of the nodes of the tree into n + 1 sets (telling which of the n players or chance takes each move);
iii. a probability distribution over the branches of each 0 move;
iv. a refinement of each of the player partitions into the the partition of information sets (which characterizes for each player the ambiguity of location on the game tree of each of his moves);
v. an identification of corresponding branches for each of the moves in each of the information sets; and
vi. a set X of outcomes and an assignment f of an outcome f(x) to each of the end points x -- or plays -- of the tree.

You will note how relations have played a role in this description: foremost as the game tree itself, but also as equivalence relations in the form of the player partition and the information partition.

In Chapter 3, when we have introduced the idea of utility, we will see how this complicated structure is translated into a far simpler mathematical structure which is much more like some of the maximization problems with which you are familiar.

2.7 ORDERINGS

In addition to relations that partition sets - - equivalence relations -- there is another important class of relations, namely those that impose an ordering on the elements of a set. We have already mentioned several examples of such relations: inclusion among subsets of a given set, greater than or equal to among numbers, and not poorer than among, say, suits of different qualities. We see that these three examples are all like an equivalence relation in being reflexive and transitive, but they differ in not being symmetric. One might be tempted, on the basis of inclusion and inequality, to suppose that orderings are inevitably anti-symmetric, for if $A \subset B$ and $A \neq B$, then we know that $B \subset A$ is false. But "not inferior to" raises some doubts about this, for a person certainly can judge two suits to be of the same quality without concluding that they must therefore be the same suit. This being the case, we are led to the following

Definition: Let R be a relation on the set A. R is called a quasi-ordering of A if it is reflexive and transitive.

Note, according to this definition any equivalence relation is also a quasi-ordering, but, of course, the converse is not so.

It seems reasonable to call a relation like "greater than or equal to" an ordering, since it orders the numbers according to magnitude, but why the prefix "quasi"? Not only relations like numerical inequality are encompassed by the definition, but also relations like set inclusion which do not manage to string things out in a single "line." Given two subsets, neither may be a subset of the other. That is to say, two elements a and b may be incomparable in the sense that neither aRb nor bRa holds. This is the reason we qualify the word "ordering" by "quasi." Another example: suppose that in some population we measure the weight and height of the people. Let x_1 denote the weight of person x and x_2 his height. Define the relation "smaller than" over the population to be: x is smaller than y if both x weighs no more than y and is no taller than y, i.e., $x_1 \leq y_1$ and $x_2 \leq y_2$. One can readily verify that this is a quasi-ordering of the people and that there may be pairs of people who are not comparable, namely: those x and y such that x weighs more than y but is at least as short as y and those such that x is as light as y and is taller than y. This kind of relation can, of course, be extended to more than two numerical dimensions, each of which is ordered according to magnitude.

It is frequently convenient to decompose a quasi-ordering into two separate relations. In the case of inequality into "strictly greater than" and "equal to." In the case of "preferred or indifferent to" into "strictly preferred to" and "indifferent to." Formally, if R is a quasi-order, we define two relations P and I as follows:

$$aPb \quad \text{if } aRb \text{ and } b\bar{R}a$$
$$aIb \quad \text{if } aRb \text{ and } bRa$$

It follows directly from the transitivity of R that both P and I are transitive. Since R is reflexive, aRa always holds; hence aIa and $a\bar{P}a$. Thus P is irreflexive and I is reflexive. Clearly, by the definition I is symmetric and P is anti-symmetric. So, in summary, a quasi-order can be decomposed in a natural fashion into a relation P, which is irreflexive, anti-symmetric, and transitive, and a relation I, which is an equivalence relation.

It sometimes happens that once we have decomposed a quasi-order into these two relations, we decide that we are willing to treat each of the equivalence classes induced by I as a unitary object. That is, we are actually interested in the relation over the set having these equivalence classes as its basic elements. There is absolutely no difficulty in defining a relation which corresponds perfectly to P over this set, for if A and B are two different equivalence classes and aPb holds for some $a \in A$ and $b \in B$, then for any $a' \in A$ and $b' \in B$, $a'Pb'$ also holds. This follows immediately from the transitivity of R. Thus, for the equivalence classes we define a new relation, call it P again, as follows: APB if $A = B$ or if, for $a \in A$ and $b \in B$, aPb. This new relation is like set "inclusion" in that it is reflexive, anti-symmetric, and transitive. In other words, it is a quasi-relation which is also anti-symmetric. Since this special class of quasi-relations is quite important, we are led to the

Definition: If R is an anti-symmetric quasi-ordering of A, it is called a partial ordering of A.

A partial order is almost the same as a quasi-order except that we can conclude that $a = b$ if both aRb and bRa hold; in a quasi-order the same condition only allows us to say a and b are "indifferent" -- which is to say, equal with respect to the property characterizing the order, but not necessarily identical elements. Set inclusion is a partial order.

While some interesting orderings do not allow us to make comparisons among all pairs of elements, others do. Examples: greater than or equal to, (optimistically) preferences people hold among commodities, etc. Presumably these are sufficiently important to be given a name.

Definition: A quasi-order R on A is called a weak ordering of A if every pair of elements is comparable, i.e., if $a,b \in A$ imply that either aRb or bRa or both holds.

Definition: A partial order R on A is called a simple ordering of A (it is also called a linear ordering and a chain) if every pair of elements is comparable.

We note that a simple order stands in the same relation to a weak order as a partial order does to a quasi-order: the former in each case being anti-symmetric, the latter not.

As we suggested above, if A_1 is a set weakly ordered by R_1, A_2 weakly ordered by $R_2, \ldots,$ and A_n by R_n, then it is always possible to induce a quasi-order on the product set $A = A_1 \times A_2 \times \ldots \times A_n$. Formally, we do so as follows: Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in A$, then we define xRy if and only if

$$x_1R_1y_1, \quad x_2R_2y_2, \quad \ldots, \text{ and } x_nR_ny_n.$$

The most familiar examples of this are when each A_i is the set of real numbers ordered by magnitude, as was the case in our example.

It is also always possible to induce a weak ordering on A, and in many contexts this type of weak order is important. Roughly speaking, what we do is order the n sets A_i according to their "importance," and then require that a more important "dimension" always have precedence over a less important one. Example: a military commander may have several courses of action, each of which will have repercussions in several quite different domains. He might evaluate them according to his potential for future action, the damage inflicted on the enemy, loss of life among his own troops, and his personal gain in prestige. If he judges these consequences to be of overriding importance in the given order, then he will always choose the course of action which makes his potential for future action greatest, but if they are all the same in that dimension he will drop to the next and choose the one which results in greatest enemy damage, but if they are also all the same on that level, he will drop down to the next, and so on. A very familiar example of this kind of hierarch of dimensions is the ordering of words in a dictionary: the first letter governs the ordering except when two words have the same first letter, in which case the second does, and so on.

In general a lexicographic ordering R of the product set A is defined as follows:

$$
\begin{aligned}
& xRy \text{ if } x_1 R_1 y_1 \text{ and } y_1 \bar{R}_1 x_1 \\
& \text{or if } x_1 R_1 y_1 \text{ and } y_1 R_1 x_1 \text{ and } x_2 R_2 y_2 \text{ and } y_2 \bar{R}_2 x_2 \\
& \text{or } \ldots.
\end{aligned}
$$

To close this section on orderings, let us append a word of caution. Throughout mathematics orderings of the types discussed are so ubquitous and useful that one tends to get a little rigid about them. This seems to be a particular problem when it comes to formulating socially and psychologically interesting "orderings" such as preferences among goods, or comparative quality of objects, etc. Much of the difficulty centers in the assumption that a preference relation, say, is transitive. One feels, somehow, that if he prefers a to b and b to c, then he should prefer a to c. It is certainly a plausible normative statement for "strict preference," but it is something else again for "preference or indifference." For imposing transitivity in the latter case implies that we are supposing "indifference" is also transitive. It is doubtful if this is often so. We have come back again to the question of discrimination which was first raised when the idea of a set was introduced. By and large people do not, and in some sense cannot, discriminate perfectly. An example may suggest the difficulty with orderings. Most people would strictly prefer a cup of coffee with one lump of sugar to one with five lumps. These same people, however, could be expected to report indifference between two cups which, no matter how much sugar they contain, only differ from each other in sugar content by a thousandth of a gram. If so, then by taking a sequence of cups from one to five cubes in increments of a thousandth of a gram, we would have to conclude from the transitivity of indifference that the person is indifferent between one and five cubes. As this is contrary to choice, we have cast doubt on the widely used assumption that indifference is transitive. To get around such dilemmas it is possible to introduce "orderings" in which P is transitive and I is not, but we shall not go into that here.

Problems

1-What kinds of relations are the following (prove your answers):

a-let "age" mean a person's age in years at his last birthday, and let the relation be "has the same age as"

b-less than one year's difference between birthdates

c-at least as tall as

d-let A be a set of cities in the United States, and let the relation be defined on $A \times A$ as the "the greater distance between two cities in scheduled airline miles."

2-Give significant industrial examples of a quasi-order, weak order, and a lexicographic order.

3-Just before the definition of a partial order, we sketched how a quasi-order induces a partial order on the set of equivalence classes of the indifference relation. We asserted that the induced relation is transitive; prove this.

2.8 FUNCTIONS

The intuitive idea of a function is widespread and of the utmost importance in almost all science. Essentially, one means by a function a rule that assigns something to each value of a variable quantity. For example, if x denotes a real number, then the function f, where $f(x) = x^3$, is the rule that assigns the real number x^3 to each value x of the variable. In addition to such power functions of algebra, many common examples of functions are known from trigonometry and the calculus: the sine, the exponential, the logarithm, etc. It is usually made clear in the calculus that we shall call any rule which assigns a real number to each value of a real-valued variable x a function. Of course, in practice attention is largely restricted to continuous functions, or at worst to those which, like $1/(1 - x)$ and $1/(1 - x)\,x$, have only one or two discontinuities.

Historically, these are among the earliest notions of a function, but during the 19th century the concept was broadened until now we have an exceedingly general and simple definition. Even in the calculus one begins to see the need to broaden the concept. For example, consider the process of taking the derivative of a function. This can be looked upon as the assignment of one function, the derivative, to another. The cosine is assigned to the sine, since $\frac{d \sin x}{dx} = \cos x$. Thus if you take the set of differentiable functions as the underlying variable, differentiation assigns to each of these another function. This is very much like our ordinary idea of a function, except that real numbers are replaced by real-valued "ordinary" functions. The sets which represent the independent and dependent variables are sets of functions, not the real numbers. But other than that, the notion is not very different. The integral, and many other operations with functions, can be viewed require special treatment.

Definition: Let D and R be two (not necessarily different or disjoint) sets. A subset F of $D \times R$ having the property that for each $d \in D$ there exists at least one $r \in R$ such that $(d,r) \in F$ is called a function from the domain D into the range R. The set

$$[r \mid r \in R \text{ and there exists } d \in D \text{ such that } (d,r) \in F]$$

is called the image of D under F. A function F is called single-valued if (d,r), $(d,r') \in F$ imply $r = r'$.

For reasons which are partly historical and partly matters of convenience, certain special notations are used for functions which differ from the usual notation of a subset of the product of two sets. Historically, the early notions of a function arose and were widely employed long before this more abstract definition was evolved, and as a result different notations were introduced. Naturally, these are better known. In addition, many of the concepts one wishes to consider about functions can be more neatly expressed in the conventional notation (see Chapter 3). There are four, somewhat different, notations which we may mention. In these F is simply the name of the function which is given by the subset F of $D \times R$; this is not the first time we have let the same symbol play two different, but closely related roles.

1-F: $D \to R$
2-F: $x \to F(x)$
3- $x \xrightarrow{F} F(x)$
4-F

But what of the most common notation of all, $F(x)$; why is that omitted? It is true that this is the most common notation, but it is misleading and, in fact, incorrect. $F(x)$ denotes the image of the point x in the domain, not the whole function which describes how each point of D is mapped into R. The function is F. We will avoid the notation F(x) for a function.

There is a certain amount of terminology about functions which it is well to have at one's finger tips. Some of it has already been introduced and used: domain, range, image, and single-valued. We say that a function is onto R if its image is R; otherwise, or if we don't know, we say it is into R. A function which is not single-valued is called multi-valued. Actually, most often one just uses the word "function" to mean "single-valued function," and prefixes it by "multi-valued" if it is not single-valued. There are a fair number of synonyms for functions, many of which have implicit conventions for their use. Among them are: mapping, transformation, and operator.

> There is a perfectly trivial way to avoid ever having to work with multi-valued functions, but often this trick does not really buy anything. Suppose F is a multi-valued function from D into R. This means that F(d) is not necessarily a single point in R, but can be a subset of R. But, of course, a subset of R is a single element in 2^R, hence we can always treat F as a single valued function from D into 2^R. The reason that this change is not always valuable can be easily illustrated. Suppose F is a real-valued function of a real variable defined as follows:
>
> $$F(x) = \begin{matrix} x \text{ for } x \geq 1 \\ 0 \text{ for } x \leq 1 \end{matrix} .$$
>
> At the point 1, $F(1) = \{0,1\}$. To make F single-valued, we would then pass from the fairly simple range of the real numbers to all possible subsets of the reals, which is an extremely complicated set having none of the neat and familiar structure of the reals. This is an awful price to pay for being unwilling to skirt about one obstreperous point in the domain.

Suppose that $F\colon D \rightarrow R$ and that $D = X \times Y$, then we say F is a function of two variables, one with domain X and the other with domain Y. If $d \in D$, then by our assumption about D, it has the form $d = (x,y)$, where $x \in X$ and $y \in Y$. Thus, $F(d) = F((x,y))$. For simplicity, $F((x,y))$ is usually written $F(x,y)$. If D is the product of n sets, then we say F is a function of n variables. For example, suppose X is the set of real numbers, then we know that $X \times X$ denotes the plane. Thus, if $F\colon X \times X \rightarrow X$, then F is an "ordinary" real-valued function of two real variables. Examples of such functions are $F(x,y) = xy$ and $G(x,y) = x + y$. So we see that the familiar multiplication and addition of numbers can be considered as functions from the plane into the real numbers.

Problems

1-The set operations of union, intersection, and complementation are all functions. Specify the domain and range of each. Are they onto or only into? Are they single- or multi-valued?

2-Prepare a list of five truly significant functions which are in one way or another involved in an industrial plant. Make at least two of them concerned with management problems. In each case carefully specify the domain, the range, and the function itself.

2.9 SUMMARY REMARKS

So far we have really done nothing; we have only introduced you to a battery of concepts which you have had to take on faith as being useful. This probably was not too difficult to do, since at least in special cases you have seen many of these notions before. Having this apparatus, we will be able to delve into its use in social science problems in the next chapter. There we shall be almost entirely concerned with the question of how to specify and to find out about functions when their ranges and domains are different from the real numbers. In the course of doing so, we will work through simple versions of several problems which have proved important in the social sciences.

But before turning to these questions, one point should be made about the ground we have covered. A number of very general ideas have been introduced, including relations, orderings, and functions in this chapter. Yet in each case it turned out that we were not required to introduce any new basic ideas. Once an idea was evolved, we were always able to formulate its definition in terms of our more primitive idea of a set. Thus, our only undefined, primitive ideas continue to be those of a set, element, and belongs to (plus the rules of logical inference). Everything else has been given meaning in terms of these primitives. This kind of economy is not only intellectually elegant, but allows us to concentrate on a relatively few primitives if later any difficulties seem to arise.

Let us turn now to the methods which have been evolved for working with functions.

CHAPTER III

AXIOMATIZATION OF FUNCTIONS

3.1 INTRODUCTION

Much mathematical work in science -- be it physical or behavioral science -- is devoted to the isolation and investigation of functions which, for one reason or another, are deemed to be of interest. In the physical sciences the major, but by no means exclusive, tools are differential equations and their various extensions and relatives which, together, are called analysis. For the most part, these methods cannot be carried over directly to the behavioral sciences because the basic behavioral variables -- at least as they are now viewed -- are not numerical. The single outstanding exception to this is economics.

This observation must not be interpreted to mean that the sets representing social and psychological variables are totally without structure; on the contrary, some structure is essential. But it just doesn't happen to be that of numbers. For most of us trained in the physical sciences, being deprived of our major weapon -- analysis -- leaves us with a bewildered empty feeling, and we tend to fall back upon non-mathematical approaches, relying on intuition and, all too often, prejudice when we have to think about problems where people are involved.

The main purpose of this chapter is to show that things are not nearly so black as they first seem and that what one is forced to do in the mathematization of behavioral problems is not, after all, so different from what one does in physics. We shall begin by talking a bit about the definition of a few familiar functions, for such definitions are not always well understood. After that we shall inquire as to just what one means by a differential equation and its solution. We will not be in the least concerned with solutions to particular differential equations, but rather with the meaning of a solution to any differential equation. Once this is understood, it will be more or less clear in principle how to extend these same ideas to domains and ranges which are not numerical. However, to stop at that point would hardly be convincing or satisfactory, and so the rest of the chapter is devoted to presenting a number of special cases with an eye to making this "in principle" extension far more concrete and meaningful. In each case, one or two very simple examples will be given to illustrate the idea, and then in starred sections more complicated examples drawn from the behavioral sciences are offered. In order to keep complications to a minimum, we have sometimes chosen formulations of these problems which are less general and less elegant than some available in the literature.

Let us stress again that there is a simple conceptual unity lying behind these examples, and that in turn there is a close conceptual -- though not technical -- similarity between them and the analytic methods you know so well. The details of presentation should not be allowed to becloud the basic simplicity and power of the axiomatic method.

3.2 DEFINING FUNCTIONS

In a way, all that we have to say in this chapter is implicit in our previous remarks about defining a set. Basically, a set must be defined via a property (sometimes presented as two or more properties for convenience of statement) which its elements, and only its elements, satisfy. There are two special cases of this which we have singled out as being of a distinctive character, rendering them almost conceptually different: First, the elements of a finite set can be listed explicitly. Second, some sets can be defined as a "combination" (e.g., union, intersection) of previously defined sets. The very same comments hold for functions, since, as we have seen in the last chapter, they are nothing more nor less than a special brand of set -- a subset of $D \times R$.

It will help, however, if we explore the implications of this remark much more fully. We will first dispose of the last two special methods of definition, and then concern ourselves in the rest of the chapter with functions defined implicitly by properties they satisfy. The most familiar examples of functions presented explicitly are those given in tabular or graphical form. Tables of the sine or the logarithm are explicit listings of two functions. To be sure, only certain selected values of the function are tabulated and then only to a certain degree of accuracy, and one must interpolate to find other values. The tabular or graphical presentation of experimental data amounts to the same thing. Indeed, any time the domain is finite, this method can be used, often to advantage. For many theoretical purposes, however, it is not suitable either because it is not sufficiently compact or because, in idealizing the problem, we have chosen an infinite set as the domain. Actually, something closely related to an explicit tabulation of a function is also possible when the domain is infinite, provided the image is finite. Example:

$$f(x) = \begin{matrix} 1, \text{ if } x \text{ is a rational number} \\ 0, \text{ if } x \text{ is an irrational number} \end{matrix} .$$

This function is given an explicit definition by relying on an implicit definition of certain subsets of the domain, in this case the definition of the rationals and irrationals. This is often a useful trick.

Definitions of new functions in terms of ones which are already known plus admissible operations (such as addition and multiplication) in the range and domain are familiar. Ordinary algebra can be thought of as being built up this way.

Another class of examples where new functions are defined in terms of old are derivatives and integrals of known functions. Sometimes they can be explicitly expressed as old functions, e.g.,

$$\frac{d \sin x}{dx} = \cos x.$$

Other times they cannot, as for example the elliptic integral

$$\int(1 - k^2\sin^2 x)^{1/2} dx.$$

Nonetheless, this integral is a well-defined function expressed in terms of an operation on an old function; for numerical work, one usually looks it up in tables.

One very general and useful construction of new functions from old is the iteration of two or more functions. Suppose f is a function from D onto R and g from R into S, i.e.,

$$\begin{aligned} f&: D \to R \text{ (onto)} \\ g&: R \to S \quad . \end{aligned}$$

We may now define a function h which is the overall effect of first mapping D into R via f and then R into S via g -- the iteration of g on f. Formally, h: D → S is defined as

$$h(d) = g[f(d)], \ d \in D.$$

It is customary to write h = g[f], or simply gf. A word of caution: suppose D = R = S = real numbers, then the symbol gf is ambiguous, for it could mean the iteration of g on f or it could mean the function H defined as follows:

$$H(x) = g(x)f(x), \ x \text{ any real number.}$$

Usually, the context will differentiate between these two meanings.

As an example of iteration, suppose D = R = S = real numbers, and $f(x) = x^2$ and $g\ (x) = \log x$, then

$$\begin{aligned} g[f(x)] &= g[x^2] \\ &= \log x^2 \\ &= 2\log x \\ &= 2g(x). \end{aligned}$$

$$\text{Note, } f[g(x)] = f[\log x] = [\log x]^2$$

Thus, in general, $f[g] \neq g[f]$.

Although defining new functions in terms of old is an extremely valuable and often not too difficult activity, it still doesn't ever get to the heart of the problem of defining functions. Somewhere that process must cease and one or more functions have either to be given explicitly or implicitly. Sometimes explicit definitions can be used, but for much theoretical work they will not do. This leaves us, then, with the major area of implicit definitions. The rest of the chapter is devoted to this.

Problems

1-Suppose f and g are defined to be

$$f(x) = b^x \text{ and } g(x) = \log_b x.$$

What can you say about $f[g]$ and $g[f]$?

2-In general, if

$$f\colon\ D \to R \text{ (onto)}$$

$$g\colon\ R \to D \text{ (onto)}$$

have the properties

$$g[f(d)] = d, \text{ for all } d \in D$$

$$f[g(r)] = r, \text{ for all } r \in R$$

we say f and g are inverses of each other. Show that a necessary and sufficient condition for a function f from D onto R to have an inverse is that for each $r \in R$, the set

$$[d \mid d \in D \text{ and } (d,r) \in f]$$

has exactly one element.

3-Let $D = \{a,b,c,d\}$ and $R = \{1,2,3\}$. Suppose $f\colon D \to R$ and $h\colon R \to D$ are explicitly defined as:

$f(a) = 2$	$h(1) = b$
$f(b) = 1$	$h(2) = a$
$f(c) = 3$	$h(3) = c$
$f(d) = 2$	

Write out $h[f]$. Restricting f to the domain $\{a,b,c\}$, write out $f[h]$.

3.3 SOME WELL KNOWN PROPERTIES OF ORDINARY FUNCTIONS

As with sets, an implicit definition of a function is a list of properties which it satisfies and which specify exactly that function -- no more, no less. Actually, in practice, we often find it convenient to discuss a whole class of functions, each of which possesses a given property. It may be worth reviewing a few of these.

In analysis, a very prevalent assumption is that a function is continuous, or at worst that it has a finite number of discontinuities. You will recall that, roughly, a real-valued function f of a real variable is *continuous at the point x* provided that whenever y is a point "near to" x, then $f(y)$ is also "near to" $f(x)$. We will not attempt to make this precise -- which amounts to making precise what we mean by "near to" -- since we shall not use continuity extensively. A function is said to be *continuous* if it is continuous at every point x. This is a property which may or may not be met by a function. For example, the function drawn in Fig. 21 and defined as:

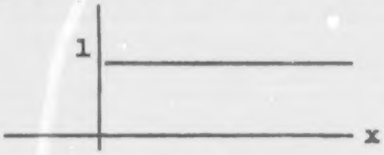


Fig. 21

$$f(x) = \begin{matrix} 0, \text{ if } x \leq 0 \\ 1, \text{ if } x > 0 \end{matrix}$$

is a step function with a single discontinuity at $x = 0$. While it is not continuous, each of its halves are. This makes it comparatively easy to work with. Not all functions have just a finite number of discontinuities, and so they are not all built up of continuous segments. For example, the function f where

$$f(x) = \begin{matrix} 1, \text{ if } x \text{ is a rational number} \\ 0, \text{ if } x \text{ is an irrational number} \end{matrix},$$

which we mentioned before, is everywhere discontinuous. Many of the functions for which you know "formulas" are continuous: e^x, x^2, $\sin x$, etc.; and others have only a finite number of discontinuities: $\log x$ where $x \geq 0$, $1/x$, $1/(1 - x)(1 + x)$, etc.

Another property which is frequently singled out in analysis is monotonicity -- whether a function is always increasing or always decreasing. Formally, we say a real-valued function f of a real variable is *monotonically increasing* if $x \leq y$ implies $f(x) \leq f(y)$; it is *strictly monotonically increasing* if $x < y$ implies $f(x) < f(y)$. The step function of Fig. 21 is monotonically increasing, but not strictly. For the domain $x \geq 0$, x^2 is strictly monotonically increasing. There are parallel definitions for decreasing functions. A number of the important functions are neither monotonic increasing nor decreasing: the sine, x^2 for all x, etc.

In one sense, each of these conditions is fairly weak, for each defines a large class of functions having that property. But there is some narrowing down. If you think only of continuous functions, the sine is included, but if you stipulate the class of continuous monotonic functions, then the sine is left out. As more and more properties are added, fewer and fewer functions can be found which meet all of them -- until, finally, you may get down to just one function meeting them or, if you're not careful, to none at all. We'll come back to this, but first let us consider the question of generalizing these two properties to a broader class of functions.

Neither the idea of continuity nor monotonicity makes much use of the properties of the number system, which means that it should not be too difficult to make them meaningful properties for functions with domains and ranges a good deal more general than the number system. Of the two, continuity is the more difficult to generalize, and as we will not need this generalization we shall do no more than say a few suggestive words about it. The only term in our informal definition of continuity which refers to properties of the number system is "near to." If we could abstract what we mean by this, then all sets having a "near to" structure on them would be suitable domains and ranges for defining continuous functions. Such an abstraction is possible, and it is known as a topology. Sets having a topology, i.e., a concept of "near to" defined on them, are known as topological spaces. So far topology has found little, if any, direct use in the attempt to do mathematical work in the behavioral sciences, and so we will not enter into it here.

The generalization of monotonicity is much more important for our purposes here, though it certainly is not nearly so important in mathematics in general. To define monotonicity, the only property of the numbers which was required was their ordering, which it will be recalled is a simple ordering. Thus, it is easy to see that the generalization can be made at least to those ranges and domains which are simply ordered. But if you look at the definition carefully, you will see that it neither matters whether the ordering is strictly antisymmetric nor whether all pairs of elements are comparable or not. We are therefore led to make the following

> Definition: Let $\not{R}$ be a quasi-ordering of the set D and $\not{S}$ a quasi-ordering of the set R. A function f with domain D and range R is said to be order preserving if for every $a,b \in D$ such that $a \not{R} b$, then $f(a) \not{S} f(b)$.
>
> (Terminological note: Some authors use "monotonic" where we have used "order preserving," but this is not very common in the literature of applications to the behavioral sciences.)

Order preserving functions are important for this reason: the image of such a function reflects the order structure of the domain, and if one knows a lot about the image, then indirectly one also knows a lot about the domain. The ordered set we know most about is, of course, the real numbers ordered by magnitude, and so it should not be surprising if at some point we attempt to map an ordered set arising out of a behavioral science problem into the real numbers. One such topic is known as utility theory (See Section 3.11). The real number system is not, however, the only ordered set we know something about (subsets under inclusion is another one), and so we should not be completely rigid about representing ordered system numerically.

So far we have introduced only properties which are not very restrictive, and this is liable to be somewhat misleading. It would appear that we would have to have very long lists of properties before we narrowed ourselves down to a single function. This is not true, and to illustrate it we will examine two properties, each of which put on very much tighter clamps. Again, let us confine ourselves to functions whose domains and ranges are the real numbers. Suppose x denotes the level of some physical variable, and f(x) some measure of the response of a system when the variable takes on the value x. Similarly, f(y) is a measure of the response when the variable has the value y. If we know both of these quantities, then do we also know the response

$f(x + y)$ when the variable has value $x + y$? Not in general. But for some systems, especially in some elementary parts of physics, the response to $x + y$ is simply the sum of the responses to x and to y separately, i.e.

$$f(x + y) = f(x) + f(y). \qquad (1)$$

This is sometimes known as the "superposition law."

What is eq. 1? Well, first of all, in technical jargon it is called a functional equation. Clearly it is an equation, but in contrast to ordinary algebraic equations which it somewhat resembles, the unknown quantity is the function f, not a number. Looked at another way, it specifies a property which must be met by those functions f which are said to solve it; it narrows down the admissible range of acceptable functions. Almost any function you can think of does not have this property. In fact, if you add to eq. 1 the condition that f be continuous, then it can be proved that

$$f(x) = ax, \qquad (2)$$

where a is any constant. It is easy to see that the functions of eq. 2 do satisfy eq. 1; it is a little more difficult to show that they are the only continuous functions which do so.

Eq. 1 plus continuity narrow us down to the functions of the simple family given by eq. 2. This is far more restrictive than anything we have seen so far. It is easy to see how, by adding a third property, we can narrow f down to a single function. All we have to do is specify the value of f at some point different from 0, for example, if we set

$$f(x_o) = f_o, \text{ where } x_o \neq 0,$$

then we see from eq. 2

$$f(x_o) = ax_o = f_o,$$

so,

$$a = f_o/x_o.$$

A unique function has been defined by three of its properties.

As a second, and somewhat similar, example, suppose we consider those functions f with domain $x > 0$, such that

1-f is continuous,

and

2-$f(xy) = f(x) + f(y)$.

In words, these are the continuous functions that map the operation of multiplication of positive numbers into the addition of numbers. From this you can guess that logarithms are included among the solutions. In fact, it can be shown (and will be later) that they are the only solutions that are continuous. If, in addition, the value of f is specified at any single point other than $x = 1$, then the base of the logarithm is specified and so the solution is unique.

The various properties of the logarithm which make it so useful and which give you the feeling that you can work with it in a way that is impossible with arbitrarily defined functions follow immediately from the functional equation $f(xy) = f(x) + f(y)$. For example, let us prove $\log x^n = n \log x$, where n is an integer. In terms of the "unknown" f, we want to show $f(x^n) = nf(x)$. The method of proof is by mathematical induction. This method is often appropriate when you have a series of related propositions, one associated with each integer. One shows by direct verification, which is often trivial, that the proposition is true for n = 1. Next, one supposes that the theorem is true for the integer n, and then establishes that this implies it is also true for the integer n + 1. These two proofs are equivalent to a proof that it is true for each n, for take any n, then the fact that it is true for 1 implies that it is true for 2, that it is true for 2 implies it is true for 3, and so on until you get to n. For our functional equation, the assertion is trivially true for n = 1. We suppose it is true for n, and attempt to show it for n + 1. From the functional equation,

$$\begin{aligned} f(x^{n+1}) &= f(x \cdot x^n) \\ &= f(x) + f(x^n). \end{aligned}$$

Substituting the induction hypothesis that $f(x^n) = nf(x)$ we find,

$$\begin{aligned} f(x^{n+1}) &= f(x) + f(x^n) \\ &= f(x) + nf(x) \\ &= (n+1)f(x). \end{aligned}$$

So, we have shown that you can get to some of the ordinary functions of analysis by an implicit definition in terms of their properties. Indeed, we would claim that this is the basic way such functions are defined, but the fact that they are very familiar and that you can use them easily in calculations tends to mask this. Anything you know about the logarithm can be derived from the two properties we have stipulated. For example, if you choose a value for the base -- a value of f for some $x \neq 1$ -- then it is possible to compute f for any other value of the argument. This is, in fact, one way to prepare a log table. We are not, of course, denying that the logarithm can be shown to be equal to a number of other expressions, which in some contexts are taken as its definition. For example, it is well known that

$$\log x = \int \frac{dx}{x} \quad .$$

To show this from our definition it is necessary to show that the expression on the right is continuous (which is trivial since all integrals are continuous) and that it satisfies the functional equation for the logarithm. Of course, it is extremely useful to know that this integral and the logarithm are the same thing, and much of elementary mathematics is devoted to such equalities. It amounts to showing that an implicitly defined function sometimes can also be defined as a combination of previously defined functions.

Problems

1-In problem 2 of the last set you showed that a necessary and sufficient condition for a function f to have an inverse is that $[d \mid d \in D, (d,r) \in f]$ is a single element set for every $r \in R$. Can you think of a simple equivalent condition (in terms of the properties defined in this section) when f is a real-valued function of a real variable? Prove your answer.

2-Using only the property $f(xy) = f(x) + f(y)$, show $f(1) = 0$.

3-Consider those real-valued functions of a real variable which satisfy the functional equation $f(x + y) = f(x)f(y)$ and are not identically 0. Show $f(0) = 1$. Can you think of any function satisfying this functional equation?

4-Use mathematical induction to show $1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$.

3.4 DIFFERENTIAL EQUATIONS

Beyond a doubt, the most familiar way to get at functions of importance in physics is via differential equations. To many, this method seems superior to all others. Certainly, it is a field which has received intense study for several hundred years and many of its results have been reduced to handbook simplicity. Although such detailed knowledge is very useful in practice, it has blinded some to the true nature of the method involved. We wish to discuss this method briefly.

Let us suppose, as an example, that we are interested in the number of radioactive atoms which have not yet decayed at time t

95

assuming that we began with N such atoms at time 0. Let this unknown number of atoms be denoted by f(t). The first important thing to recognize is that we have no immediate precise grasp on the function f. Our intuition tells us that if N is very large, then we will not get into serious trouble if we suppose f is continuous. Furthermore, the very concept of radioactivity insures us that f will be a monotonically decreasing function of time, but beyond that we have little immediate feel for f itself. The second important point is that, while we do not know f directly, we can say certain things about f and functions of f from physical principles. In particular, it can be shown that f has the property that its rate of change in time is proportional to its value, i.e.,

$$\frac{df}{dt} = -kf, \text{ where } k > 0.$$

It is well known that this differential equation is solved by

$$f(t) = A\exp(-kt),$$

where A is a constant. Now, introduce the initial condition

$$f(0) = N,$$

then we get the unique solution

$$f(t) = N\exp(-kt).$$

All of this is trivial and very well known. But suppose you place yourself in the shoes of the scientist who first arrived at such an equation from some physical process (certainly not radioactivity, but that is of no matter). You know nothing of the theory of differential equations, for it has not yet been formulated, but nevertheless you have posed this problem: to find those functions f of time which satisfy the following properties:

$$\frac{df}{dt} = -kt$$

$$f(0) = N$$

(Note: by writing the first equation, we implicitly impose the condition that it is meaningful, i.e., that f is everwhere differentiable. It is well known that this implies that f is a continuous function of time.) Given this problem, what would be the first question you would ask? If your answer is "what is the solution" or some variant of it, then we doubt that you have really placed yourself in this early scientist's shoes. You know that there is a solution and you would want to find it. He, however, would not have known immediately from the two properties he had written down that a solution exists. It is not, when you think about it, completely obvious that these two peculiar conditions are necessarily satisfied by any function f. For instance, had he set up the problem to find those f's which satisfy

$$\frac{df}{dt} = -kf \quad ,$$

$$\frac{d^2f}{dt^2} = ct$$

then he could not have found a solution.

No, his first task would be to assure himself that he had in fact posed a soluable problem -- to show the existence of a solution. To be sure, he might do this by demonstrating that the exponential is a solution, but what if he did not have any hint that this was so? It can be quite futile to try randomly one function after another. Furthermore, he might realize that his was but a special case of a whole class of similar problems, and so he might be tempted to use indirect methods to show once and for all the existence of a solution for each member of the class of differential equation problems. In fact, this is the reason that you don't have to worry about the existence of solutions to most of the equations which arise in physics and engineering. It has been done for very general classes of differential equations. Advanced courses in differential equations are very largely concerned with such questions.

Once having established to his satisfaction that the problem posed is solvable, then one might suppose that his next task would be to express the solution in terms of known functions. But again, this is doubtful. By what stretch of the imagination could he suppose that there is only one function satisfying the conditions he has posed? This is not to deny that we know that there is only one solution to a first order linear differential equation plus one initial condition, but is it obvious from just looking at the two conditions? Thus, you might expect that his second task would be to show that the solution is unique. Again, this is something which can be done without knowing the functions which solve the problem, and it can be done once and for all for broad classes of differential equations. It is because this has been done in the mathematical literature, and not because it is unimportant, that the engineer and physicist need not be much concerned today with uniqueness problems.

In any characterization of a function by properties which it must satisfy -- differential equations are one special case -- the two questions of extence and uniqueness are of primary importance. In the applications of mathematics to the behavioral sciences which we shall mention, it will be necessary for us to deal explicitly with these questions, for we do not have a comprehensive general theory of existence and uniqueness of functions on which to fall back. Once we have done this, then we may want to worry about describing the function in some other terms. This is what a solution to a differential equation is. It is customary to say that we have solved a differential equation if we can express the solution in the form of integrals of known functions. Outside that realm, the criteria for what constitutes an acceptable representation of a solution are far vaguer, and a certain amount of judgment is needed. In some cases, we are satisfied with a demonstration of existence. In others we need to have practical methods to find certain values of the function.

To return to our hypothetical mathematician, once assured that a solution exists and having begun to worry about uniqueness, he might well conjecture that it is not unique. Such a conjecture would be correct if at first he only set up the differential equation without the initial condition. Any function of the form Aexp(-kt) is a solution to the differential equation alone. This is, of course, well worth knowing, for it tells you that all solutions are fundamentally the same shape and differ only by a scale factor. Put another way, the ratio of two different solutions to this differential equation is some constant. Not only is that a compact statement of the situation described by the equation, but it also suggests the nature of the condition which has to be added to render f unique.

Thus, if you set up some properties to describe one or more functions and you find more rather than one, then the next thing to do is to try to find out how two different solutions are related to each other. This means that you want to describe the transformations which map one solution into the others. Ideally, but not invariably, the class of transformations which describe how to go from one solution to the others has certain nice closure properties: any transformation from the class maps a solution into a solution, and any two solutions are related by one of the transformations in the class. In this case, the mathematician's problem is to give a compact description of this class of transformations and of one of the solutions.

In summary then, our mathematician first working on this differential equation problem would have had to do the following things:

1-show that a solution to the problem exists,
2-determine whether it is unique or not, and
3-if it is not unique, ascertain how the several solutions are related to one another.

There is nothing about these three steps which rests upon the fact that his was a differential equation problem. The same questions are meaningful and important whenever a function is defined implicitly by properties which it must meet. As we shall see in the following sections, some of the terminology is a little different when we work in a more general context, but the ideas are not different.

3.5 AXIOMATIZATION AND INCONSISTENCY

The first question when a function is implicitly defined in a context different from the real numbers is: what context? The mathematical nature of both the domain and range of the function must be described. Usually, this becomes apparent from the verbal formulation of the problem in question and from the aims of the analysis. Nonetheless, it is an extremely important step which determines to a surprising degree the success or failure of the effort. For example, it has sometimes happened that an apparently intractable or very messy problem is made very much simpler by an appropriate change in the domain or the range. In any event, these two things have to be specified precisely at the outset. (By and large, in engineering problems both the domain and range are the real numbers and usually this is taken for granted without any comment.)

Once that is done, then appropriate properties of the function must be specified. Without a doubt this is the most difficult part of the task. It is a subtle art, requiring both a considerable sophistication in mathematics and a perceptive understanding of the physical or behavioral situation one is trying to abstract. Mathematical experience is needed so that the problem formulated will lead to suitable and interesting mathematical results, and the substantive problem must be well grasped so that the mathematics reflects it rather than some other problem. The interplay between these two demands is most tenuous and only rarely is there a fruitful union. For the traditional problems of physics, it now seems comparatively simple to set up such conditions, for there are the known laws of physics to be drawn on. So far there are precious few laws, or even hints of laws, in the non-physical sciences, and one is forced to considerations which differ considerably from those used in physics. Among these are: unconfirmed or only partially confirmed guesses as to the laws operating, assumptions as to the statistical independence of two processes, ethical and normative demands, and *a priori* demands on the nature of scientific measures. Some of these will be illustrated in our later examples.

In any event, a list of properties of the unknown function is presented. When speaking in the general context, these properties are known as *axioms*, and the set of them as an *axiom system* for the unknown function. The whole activity is called axiomatizing a function or giving an axiomatic definition of a function. While such terminology is rarely appled to the mathematics of physics, we could say that the differential equation and its initial condition mentioned in the last section are each axioms which together characterize a specific exponential function.

The first problem, as we said, to be posed of any axiom system is existence: does there exist a function satisfying all the axioms. Again, in the general context, the language is a bit different. Often one does not speak of the existence of a solution to the axiom system, but rather of the "consistency" of the axioms. Also one speaks of a function "satisfying" them. If there is no inner contradition among the axioms, i.e., if there exists a function which satisfies all of them simultaneously, then the axiom system is said to be consistent. If they are contradictory, they are said to be inconsistent.

A trivial example of an inconsistent system: Suppose $D = \{1,2,3\}$ is simply ordered by magnitude and $R = \{1,2\}$ is also simply ordered by magnitude. Problem: to find any functions f: $D \rightarrow R$ satisfying the axiom

$$\text{if } x,y \in D \text{ and } x < y, \text{ then } f(x) < f(y).$$

It is easy to show that no such function exists -- that the axiom is inconsistent (with the domain and range) -- for by two applications of the axiom we have

$$\begin{matrix} 1 < 2 \text{ so } f(1) < f(2) \\ 2 < 3 \text{ so } f(2) < f(3) \end{matrix},$$

hence the range has three distinct points, contrary to assumption.

A less trivial example: When we introduced the concepts of union, intersection and inclusion of sets, we drew some parallels with ordinary addition, multiplication, and inequality. We already know that the parallel is far from perfect, but it would be interesting to see how far it goes. The following problem sets up one of the simplest analogies that might be possible. Suppose U is a set with two or more elements, $D = 2^U$, and R = real numbers. To find those functions F: $D \rightarrow R$ such that the following axioms are met:

1-if $A,B \in D$ and $A \cap B = \phi$, $F(A \cup B) = F(A) + F(B)$;
2-if $A,B \in D$, $F(A \cap B) = F(A)F(B)$; and
3-if $A,B \in D$ and $A \neq B$, then $A \subset B$ implies $F(A) < F(B)$.

We claim that this axiom system is inconsistent. A proof goes as follows. From axiom 1,

102

$$F(U) = F(U \cup \phi) = F(U) + F(\phi),$$

hence $F(\phi) = 0$. For any $A \in D$, axiom 2 states

$$F(A) = F(A \cap A) = F(A)^2,$$

so $F(A) = 0$ or 1. If $A \neq \phi$, axiom 3 implies $F(A) = 1$. But since U has two or more elements, there is at least one subset A different from U and from ϕ, so $A \subset U$ and $F(A) = 1 = F(U)$, which is impossible by axiom 3. Thus the system is not consistent.

Although the first example was completely trivial, and the second one not very difficult, they both illustrate the basic procedure involved in showing an axiom system to be inconsistent. Our next section is devoted to a more complicated and interesting inconsistent axiom system. It arose in welfare economics, and it is interesting mainly because one does not at first suspect the axioms to be inconsistent.

Problems

1-In the last example, suppose we drop axiom 2. What function satisfies axioms 1 and 3?

2-Let D and R be the real numbers, and $f\colon D \to R$. Show that the following three axioms are inconsistent:

i. $f(xy) = f(x) + f(y)$,

ii. $f(x + y) = f(x)f(y)$.

iii. f has at least two different values.

(Note: do not assume f is continuous.)

*3.6 THE ARROW SOCIAL CHOICE PROBLEM

Roughly speaking, the concept of a fair social decision is one which is arrived at by taking into account equally the preferences of each (adult) individual for the several alternatives that have arisen or that are presented by the leaders of the society. It amounts to a rule that enables society to pass from the "votes" of the individuals to a social decision, but not just any rule. The rule must be "fair;" it must "take into account equally" the preferences of each of the individuals. The problem is to agree upon what we mean by "fair." What properties characterize a fair rule? Once we have agreed upon that, then we can investigate mathematically those rules which satisfy these properties and we can ask whether particular rules in common use meet them. If not, then where do they fail and is this important? For example, the rule most commonly used in Western societies is simple majority rule. Does it satisfy our intuitive ideas of fairness?

Such problems have been discussed for some years in the literature of political science and welfare economics, but not until 1951 was one formulated and attacked as a mathematical problem. This work, and a sketch of the background of the problem, can be found in K. J. Arrow's Social Choice and Individual Values, John Wiley and Sons, New York (1951). This important book attracted a good deal of attention and resulted in not a little controversy and misinterpretation; as a result there have been a dozen or so journal publications on the problem since then. The formulation we shall present, which in many ways is simpler than Arrow's presentation, is based upon one of these papers: Weldon, J. C., "On the Problem of Social Welfare Functions," Canadian J. Econ. and Pol. Sc., 18, 1952, 452-463.

We begin with two known sets. First, the finite set A of m distinct alternatives presented to society. Second, the finite set I of n individuals composing the society, which may be as small as a three-man committee or as large as the whole electorate of a nation. We shall suppose that each of the individuals "votes" on the alternatives in A and these votes are entered into a "machine," which will be described below, out of which comes the social decision. We must now translate into mathematical terms what we shall mean by "vote" and by "machine."

We shall not mean by "vote" what you expect -- the selection of the most preferred alternative. Rather, we shall suppose that each person orders the whole set A according to his preferences. This gives a good deal more information about his preferences than just the selection of his most preferred alternative, and it should be desirable to utilize this added information in one way or another when reaching a social choice. Now, what do we mean when we say he orders the alternatives? We take this to mean that each person rank orders them, i.e., he states which alternatives is most preferred (say, by ranking it 1), which is next most preferred (by ranking it 2), etc. We will also allow him to report that he is indifferent between pairs of alternatives if he chooses. In terms of our previous terminology, the reported preference orderings will be weak orderings of A (see section 2.7). Thus, for example, if A = {a,b,c,d}, then an admissible preference ordering is for a person to say he prefers b to a,c, and d, a to c and d, and that he is indifferent between c and d. In other words, he rank orders them b,a,c - d. The following preference pattern, however, is not admissible: a is preferred to c and d, b to a and c, c to d, and d to b. It is not admissible because it is a non-transitive relation: a is preferred to d, d to b, and b to a.

On any given set A a number of different weak orderings are possible, the number increasing very rapidly as the size of A increases. Nonetheless, we shall have to deal with all of the possible weak orderings of A; let us call this set of weak orderings W. Our supposition then is that each person selects exactly one element from W, i.e., he votes in the usual sense of the word for just one of the weak orderings in W, and this selection is fed into the "machine" for making social decisions.

This leads us to the second question, what do we mean by the word "machine" in this context. Whatever its detailed physical realization, it must have this property: for any selection of n weak orderings from

W, one by each individual, it comes up with a social decision based upon them. That is to say, it is a function whose independent variables are the weak orderings selected by individuals and whose dependent variable is the set of possible social decisions. We must make this more precise.

First, let us consider the independent variable -- the domain -- of this function. When the individuals each select a weak order -- say, 1 chooses R_1, 2 chooses $R_2, \ldots$, n chooses R_n -- then the whole society has chosen an n-tuple $(R_1, R_2, \ldots, R_n)$, where each $R_i \in W$. In other words, the whole society has chosen an element from the product set

$$\mathcal{W} = W \times W \times \ldots \times W \text{ (n times).}$$

Since each individual is free to choose any of the weak orderings of A, i.e., any element of W, the society as a whole can select any element from the product set $\mathcal{W}$.

Once society has selected an element from the product set $\mathcal{W}$, then the role of the "machine" is to reduce this complex of information into some sort of decision about the alternatives in A. At the very least, the machine must transform such an element of $\mathcal{W}$ into an element of W, i.e., into a weak ordering of the elements of A. Thus, we take the range of the function to be W. Any function with domain $\mathcal{W}$ and range W will be called a social function.

In summary, then, the framework of our problem is this: A set A with m elements and a set I with n elements are given. W is the set of all weak orderings of A and $\mathcal{W} = W \times W \times \ldots \times W$ (n times). A function F: $\mathcal{W} \rightarrow W$ will be known as a social function. For convenience, we shall denote by R the generic ordering selected by the social function F, i.e.,

$$R = F(R_1, R_2, \ldots, R_n).$$

You will recall that in section 2.7 we showed how any weak ordering R can be broken down into a "strict preference" ordering P and an "indifference" relation I, which were defined as follows:

aPb if and only if aRb and $b\bar{R}a$

aIb if and only if aRb and bRa.

If R_i denotes a weak ordering, the corresponding P and I relations will be denoted by P_i and I_i.

Now that we have set up the general framework of the problem, our next task is to arrive at conditions on the function F such that it can be called "fair," This will be most easily done by introducing an auxiliary concept defined in terms of F, and then stating the conditions of fairness in terms of this concept. We suppose that F is a fixed social function and we let V denote a subset of individuals, i.e., $V \subset I$. If a and $b \in A$, then the set V can be considered decisive for alternative a against b provided that whenever all the members of V prefer a to b and everyone outside V prefers b to a, then society prefers a to b. Stated formally, the subset V is said to be decisive for a against b if F has the property that if

aP_ib, for all $i \in V$

and

bP_ia, for all $i \in \bar{V}$

then

aPb.

We shall now formulate the four conditions of "fairness," the last three being in terms of decisive sets. The first three are not particularly controversial and will be relatively easy to agree upon, but the last one will require more discussion.

The first axiom only requires that both the set of alternatives and the set of people be adequately large to have an interesting problem.

Axiom 1. The sets A and I shall each have three or more elements, i.e.,

$$m \geq 3 \text{ and } n \geq 3.$$

The second axiom simply says that whenever there is unanimity in society between a pair of alternatives, then the social function shall reflect this unanimity.

Axiom 2. For any pair of alternatives, the set I is decisive.

Our third requirement of fairness reflects the generally accepted belief that there should not be a dictator -- a person whose preference alone for one alternative over another commands society to have the same preference.

Axiom 3. If $i \in I$, then $\{i\}$ is not decisive for any pair of alternatives.

The next and last condition will bear considerable more discussion before we state it, for it is much the strongest and most controversial of the axioms. Suppose the n weak orderings $(R_1, R_2, \ldots, R_n)$, one from each individual, leads to the social ordering R according to a given social function F. Let us now focus on two alternatives a and b, and let us suppose that society prefers a to b. The last assumption does not really result in any loss of generality, for we can always interchange the labels on the alternatives. Within society, a certain set of people will have stated that they prefer a to b, another set that they are indifferent between a and b, and the remainder that they prefer b to a. Among the other possible pairs of alternatives, each individual will have reported some pattern of preference. Question: should society ever change its preference for a over b if all the members keep their preferences between a and b, but alter some (or all) of their preferences among the other alternatives? Put another way, if the individual preferences between a and b are held fixed, but those among other, irrelevant, alternatives are changed, should there ever be any change in the social decision concerning a and b?

The major argument for answering No is this: If the social decision between a and b also depends upon how individuals order the other alternatives, it may well be worthwhile for an individual to misrepresent his true preferences in order to put extra weight on an alternative about which he feels strongly. Just how he should misrepresent his preferences in order to emphasize a particular alternative will depend upon the function F and upon the preferences expressed by the other members of society. Thus, one enters into the complex domain of strategic considerations where decisions depend upon estimates of what other individuals are going to do. To be sure, this is not a real objection when n is extremely large, but in small committees it can become a serious problem. For this reason, it is argued, the social function should have the property that it is independent of irrelevant alternatives. We state this formally as

Axiom 4. If, for some $(R_1, R_2, \ldots, R_n)$, F has the property that

$$\begin{aligned} &aP_i b \text{ for } i \in V, \\ &bP_i a \text{ for } i \in \overline{V}, \text{ and} \\ &aPb, \end{aligned}$$

then V is decisive for a against b.

Any social function satisfying axioms 1 through 4 is called a social welfare function. Arrow's principle result, known as his impossibility theorem, states that there is no social welfare function, i.e.,

Theorem. Axioms 1 through 4 are inconsistent.

Proof. We shall suppose that a social welfare function exists, and then show that this leads to a contradiction. Specifically, we shall use a "downward" mathematical induction, starting with the set I, to show that there must exist a single element decisive set, which is contrary to axiom 3. To do this, two steps are involved. First, we must establish that the set I is decisive, but this is assured by axiom 2. Second, we must show that if there is a decisive set with q elements, where $2 \leq q \leq n$, then this implies there is a decisive set with q-1 elements. Together, these two statements imply that there is a decisive set with one element, which violates axiom 3. So to prove the theorem, we need only prove the induction step.

Suppose V_q is a set with q elements which is decisive for some element a against some element b of A. Let α be any element of V_q, and define $V_{q-1} = V_q - (\alpha)$. $V_{q-1} \neq \emptyset$ since $q \geq 2$. According to to axiom 1, there is at least one element in A different from a and b; let c be one such. We will now show that V_{q-1} must be decisive for a against c, which will prove the induction step and so the theorem.

Consider any n-tuple of weak orderings $(R_1, R_2, \ldots, R_n)$ which include the following strict preferences:

i. aP_ib, bP_ic and aP_ic for $i \in V_{q-1}$,
ii. $cP_\alpha a$, $aP_\alpha b$ and $cP_\alpha b$,
iii. bP_jc, cP_ja and bP_ja for $j \in \overline{V}_q$.

Define $R = F(R_1, R_2, \ldots, R_n)$. We claim that aPb and bRc. First, we have assumed that V_q is decisive for a against b and we have chosen the orderings R_i such that aP_ib for $i \in V_q$ and bP_ja for $j \in \overline{V}_q$, so aPb. To show bRc, we suppose this is not the case, i.e., cPb. But observe that by choice of the R_i,

$$cP_\alpha b \text{ and } bP_ic, \text{ for } i \neq \alpha,$$

110

hence according to axiom 4 this would mean $\{\alpha\}$ is decisive for c against b. As this violates axiom 3, we must conclude that bRc.

Since R is a weak ordering, aPb and bRc imply aPc. But, by choice of the R_i,

$$aP_i c, \text{ for } i \in V_{q-1}, \text{ and } cP_i a, \text{ for } i \in \overline{V}_{q-1}.$$

Thus, axiom 4 implies that V_{q-1} is decisive for a against c, as was to be proved.

As we pointed out earlier, the truth of the induction step implies the existence of a single element decisive set, which is impossible. Thus, our original assumption that a social welfare function exists is untenable, and the theorem is proved.

Many people have found this result disconcerting because they have been willing to agree to each of the four conditions as necessary requirements of "fairness" and at the same time have felt certain that a "fair" machine (function) could be devised. The fact of the matter is that this is not so. Once convinced, some emotionally reject the whole process and head to other activities; others become intrigued with the question whether the problem has been unfortunately formulated and whether some modification might not result in positive results.

Such research is currently going on and some positive results have been obtained. Although we cannot go into this work, the three major directions it has taken can be indicated. The first is to question the whole formulation of the problem in terms of weak orderings. Basically this amounts to a total recasting of the problem. The second direction rests upon the empirical observation that in any given culture it is unlikely that all possible combinations of preference patterns will ever arise. There are usually strong correlations among the weak orders registered by the members of the same society, and so in practice we are asking too much when we demand that F be defined over the whole of $\mathcal{V}$. It will generally suffice to know F for some subset of $\mathcal{V}$. The tricky task is to choose a suitable subset: one that seems to include all cases which arise empirically and, at the same time, leads to a mathematically tractable problem. There has been some success in this direction.

The third major tact is to drop the condition of the independence of irrelevant alternatives (axiom 4). This permits the participants to enter into strategy considerations when reporting preferences, but in many contexts this does not really seem relevant or important. However, it is not sufficient just to drop axiom 4. It must be replaced by some other condition, for one can easily produce examples of functions meeting the first three axioms which are impossible to consider "fair" social functions (see problems 2 and 3 below).

***Problems**

1-In a democratic society it is often claimed that majority rule is a "fair" method to reach social choices. The function F representing majority rule is defined as follows:

aPb if and only if $[i \mid i \in I \text{ and } aP_i b]$ has more elements than $[i \mid i \in I \text{ and } bP_i a]$.

aIb if and only if these two sets have exactly the same number of elements.

Arrow's theorem asserts that this function cannot be "fair" in the sense of being a social function which meets axioms 1 through 4. Where does it fail? Prove your answers (examples of violations will suffice).

2-Let $n = 3$ and let $A = (a_1, a_2, \ldots, a_m)$. Let S denote the weak ordering of A in which a_1 is strictly preferred to a_2, a_2 to a_3, etc. Let S* denote the converse ordering where a_m is strictly preferred to a_{m-1}, a_{m-1} to a_{m-2}, etc. Let F be defined as follows:

$$F(R_1,R_2,R_3) = \begin{cases} R_1, \text{ if } R_1 \neq S \text{ or } S^* \\ R_2, \text{ if } R_1 = S \text{ or } S^* \text{ and } R_2 \neq S \text{ or } S^* \\ R_3, \text{ if } R_1 = S \text{ or } S^* \text{ and } R_2 = S \text{ or } S^*. \end{cases}$$

Show that F satisfies axioms 1, 2, and 3.

3-Let $F^{(3)}$ denote any function satisfying axioms 1, 2, and 3 for a given A and n = 3 (by problem 2, at least one such function exists). Show that for the same set A and n >3, the function $F^{(n)}$ defined below also satisfies the first three axioms:

$$F^{(n)}(R_1,R_2,\ldots,R_n) = F^{(3)}(R_1,R_2,R_3).$$

3.7 CONSISTENCY AND UNIQUENESS

In general, one does not try to construct inconsistent axiom systems, for one is usually trying to get at functions which one is pretty sure exist. Proofs of inconsistency serve primarily to show either that one's intuition has been in some way faulty and that the axiomatization does not really capture what was intended, or that what people have been talking about is non-existent. These are both important services, but they give neither the author nor the reader the same constructive satisfaction as yielded by a positive result.

To show that an axiom system is consistent, two general methods are available. Either one can exhibit a function which satisfies the axiom system, or one can devise a proof which shows that there must be such a function even though one is not explicitly produced. Simple examples of the indirect method are not easily come by; at least we have not thought of one. The direct method assumes one of two forms. First, the axioms can be manipulated in such a way as to derive a necessary mathematical form for any function satisfying them, and then it is shown that this function (or functions) does in fact satisfy the axioms. It is important that this last step be carried out, for from an inconsistent set of axioms one can sometimes derive a necessary functional form, which nonetheless cannot satisfy them. There is a tendency to forget to make this verification. Second, a function may be produced (usually preceded by the phrase "Consider the following...") and then it is shown that it satisfies the axioms. Often such functions are complicated, and one wonders from whence they spring. Usually, the author is hard pressed to say; he will have drawn up on his mathematical experience, or had some insight into the problem, or tried other functions and gradually modified them into the correct one, but, above all, he will have had some luck. There is no set of rules that can be set down.

As an example, suppose the question is raised whether the functional equation

$$f(xy) = f(x) + f(y),$$

where f is a real-valued function of a real variable, has a solution. It will suffice to produce one. Consider the function f, where

$$f(x) = \int_1^x \frac{dt}{t} .$$

Observe that if the change of variable u = ty is made, then

$$f(x) = \int_y^{xy} \frac{du}{u} ,$$

so

$$\begin{aligned} f(x) + f(y) &= \int_y^{xy} \frac{du}{u} + \int_1^y \frac{du}{u} \\ &= \int_1^{xy} \frac{du}{u} \\ &= f(xy). \end{aligned}$$

Thus, the given integral is a solution to the functional equation.

In whatever way existence has been established, once it is done then questions of uniqueness arise. For instance, is the solution given above the only one to that functional equation, or are there others? The answer is that there are others, but as we shall see they must be discontinuous. Roughly, there are two general procedures to establish uniqueness. One can manipulate the axioms to derive the necessary form of the function, and if this necessary form is unique, then we know that if there is any function satisfying the axioms at all, it must be unique. As we stated above, one must actually show that the necessary form for the solution is in fact a solution. The other method, which we shall illustrate, begins with the assumption that there are two solutions and shows that they must actually be the same. One does not need to have an explicit representation of any solution to the problem to use this method, which is often an advantage.

Suppose f is a real-valued function of a positive real variable which satisfies the following axioms:

Axiom 1. f is continuous for $x > 0$,

Axiom 2. $f(xy) = f(x) + f(y)$,

Axiom 3. $f(x_o) = f_o$, where f_o is a constant and $x_o \neq 0$ or 1.

We show that f is unique. First, we show that any function satisfying axioms 1 and 2 has the property that $f(x^y) = yf(x)$, for any real and positive y. If y is an integer, n say, then from section 3.3 we know that $f(x^n) = nf(x)$. Now, suppose we let $x = z^n$, then

$$f(x) = f(z^n) = nf(z),$$

115

so

$$f(z) = f(x^{1/n}) = \frac{1}{n}\ f(x).$$

Thus, if m and n are integers, these two results combine to show

$$f(x^{m/n}) = \frac{m}{n}\ f(x).$$

But it is well known that if y is any real number, then we can choose integers m and n such that m/n is arbitrarily close to y. Thus, by the continuity of f, it follows that

$$f(x^y) = yf(x).$$

Now, let us suppose that f and f' are any two functions satisfying axioms 1, 2, and 3. We show f = f'. Any point x can be expressed in the form $x = {x_o}^y$ for some appropriate $y > 0$, so

$$\frac{f(x)}{\text{[illegible]}x)} = \frac{f({x_o}^y)}{f'({x_o}^y)}$$

$$= \frac{yf(x_o)}{yf'(x_o)}$$

$$= \frac{f_o}{f_o}$$

$$= 1.$$

Thus, f = f', or, in other words the function satisfying the three axioms is unique.

Two examples of consistent axiom systems which are satisfied by a unique function and which have arisen in the behavioral sciences are given below. The derivation is given only in the first case.

*3.8 THE INFORMATION MEASURE

A postwar development that has attracted a good deal of attention is the mathematical theory of communication. It arose out of electrical communication problems, where it has been widely used and elaborated, and its basic formulation is due largely to the work of Wiener and Shannon. A standard reference is: Shannon, C. E and Weaver, W. The Mathematical Theory of Communication, University of Illinois Press, Urbana (1949). Largely because it really is a (special) theory of statistical inference, it has had considerable impact outside the area of electrical communication, especially in psychology. For a fairly comprehensive survey of both the theory and its applications to psychology, see: Luce, R. D., A Survey of the Theory of Selective Information and Some of its Behavioral Applications, Technical Report No. 8 (revised), 1956, Behavioral Models Project, Columbia University.

We cannot go into any of the details of the theory here except to derive the mathematical form of the central function employed. This function is interpreted as a measure of the "average amount of information transmitted" by messages in a communication system, where, however, these words have a meaning which, though reasonable, is somewhat different from common sense usage. The measure is often called the "(average) amount of information transmitted," but the shorter labels "entropy" (because of its formal identity to the expression for physical entropy in statistical mechanics) and "uncertainty" are widely used.

Consider the following idealization of many communication systems. A set A of n alternatives is given, and from it messages are formed by successive temporal selections (with replacement). For example, A might represent an alphabet and the successive selections are used to form words and sentences. Such messages are then transmitted through a communication system, which involves certain physical components, to its destination. Let us consider the arrival of such a message from the point of view of the person (the destination) receiving it. He is rarely, if ever, certain what symbol he will next receive, for if he were completely certain it would be pointless to transmit it. No information will be conveyed when the receiver is able to predict with certainty what he will receive. But one should not jump from this to supposing that he must be completely uncertain as to what he will receive. If the person is sending the message in English, then the receiver knows a priori that the probability of receiving an "e" is a good deal larger than the probability of receiving a "z". Such knowledge is known to everyone speaking English, and it is continually employed when inferring the symbols sent via a channel which introduces some distortion, as in noisy telephone communication.

Thus, in general, we can suppose that there is a known probability distribution over the elements of A which describes the a priori probability that each is selected. Suppose that A = {1,2, ... n}, where the numbers are simply labels for the elements in A. The probability distribution is then some set of numbers

$$p(1), p(2), \ldots, p(n)$$

with the properties

$$p(i) \geq 0, \text{ for } i = 1,2,\ldots, n$$

and

$$p(1) + p(2) + \ldots p(n) = 1.$$

But this is hardly enough to describe the statistical structure of most sources of messages. For example, in any natural language the selections made are not independent of each other. You know in English that if you receive a "q," the probability is pretty close[*] to 1 that the next letter will be "u", even though "u" has a very low a priori probability of occurring. We will, however, ignore this observation and make the assumption that we are working with a source in which successive selections from A are statistically independent. That is to say, the probability of selecting element $i \in A$ is $p(i)$ no matter what has preceded it. No doubt this appear to be an excessively restrictive assumption since it seems to eliminate all natural languages from consideration, but in point of fact it turns out that it is easy to extend the information measure for independent selections to non-independent selections. For example, if the dependence extends only to the immediately preceding symbol, then we work with the joint probability distribution $p(i,j)$ defined over the product set $A \times A$. But this is still a probability distribution over a set, albeit a special set, and so it will have been included in our study of the independent case.

As we suggested before, no information can be transmitted if one of the symbols in A is certain to occur, i.e., if $p(i) = 1$ for some $i \in A$. Furthermore, as the symbols become more and more equiprobable, then more and more information can be transmitted. For example, if A has only two elements, 1 and 2, and if the probability of 1 being selected is 0.9, then more information is transmitted on the average then when the probability is 1.0, i.e., when none is transmitted. But suppose the probability is dropped down to 0.8, then isn't it more revealing to receive the symbol 1 than if the probability had been 0.9? Extending this argument you see that for two symbols the maximum information must be transmitted when each has probability 0.5, i.e., when they are equiprobable. Is it suitable to take the probabilities as a measure of the information transmitted? And if it is, what probability or combination of probabilities should we employ when A has three or more elements? Since any proposal one might make would seem totally ad hoc, one is led to consider reasonable properties for such a measure of the average amount of information transmitted and to see whether or not this singles out a particular measure. This we do.

---

[*] Words such as "Iraq" prevent it from being exactly 1.

First of all, since we are speaking of a measure, we mean a function which has the real numbers as its range. Second, since we have spoken of it as a measure of the average amount of information transmitted, we presumably mean that we shall find a value of the function for each alternative and then take the average of it over all the alternatives in A. But each alternative i is characterized by its probability p(i) of occurring, so the function depends upon that. So the measure is a real-valued function f with domain the real interval from 0 through 1. Already we have made a very strong assumption, one which is similar to the condition of the independence of irrelevant alternatives in the Arrow social choice problem. We have not only said that the measure of information transmitted by a selection of the symbol i depends upon p(i), but also that it depends only upon p(i). The distribution of probability over the other alternatives is completely irrelevant!

Accepting this, we now introduce three axioms. First, it seems plausible that if p(i) is changed only slightly, then the amount of information transmitted should also change only slightly (though not necessarily proportionately). Thus, we impose

Axiom 1. f shall be a continuous function of p(i).

Suppose that two successive selections are made from A, say i and then j. Since we have assumed that selections are statistically independent, we know that the probebility of the joint occurrence of i and then j, (i,j), is simply given by the product of the probabilities of the individual occurrences taken separately, i.e.,

120

$$p(i,j) = p(i)p(j).$$

Furthermore, given that the selections are independent, it seems plausible that the total amount of information transmitted is simply the sum of the amount transmitted by i and the amount transmitted by j. This is reasonable only because they are independent. We would not want to say that the amount of information transmitted by (q,u) in English is very different from that transmitted by q alone -- certainly it is not as much as the sum of q and u taken separately. But since we have assumed independence, it is reasonable to impose

Axiom 2. $f[p(i)p(j)] = f[p(i)] + f[p(j)]$.

Finally, in any measurement problem it is necessary to agree upon some unit in terms of which measurements are made: centimeters for length, seconds for time, grams for mass, etc. In this field it has proved convenient to say that one unit of information has been transmitted whenever a selection occurs between two equally likely alternatives. The unit is called a bit. Thus, we have

Axiom 3. $f(1/2) = 1$.

These three axioms should look familiar, they are the same as those we set up in the preceding section. So we know that there is a unique function satisfying them, which must be the logarithm to some base which is determined by axiom 3. It is easy to see that it is the base 2, i.e.,

$$f(p) = -\log_2 p.$$

Now if we take expected values over all the elements in A, we obtain Shannon's famed expression for the average amount of information transmitted:

$$-p(1)\log_2 p(1) + p(2)\log_2 p(2) + \ldots + p(n)\log_2 p(n) = -\sum_{i=1}^{n} p(i)\log_2 p(i).$$

In terms of this concept and two others -- channel capacity and noise -- Shannon was able to prove some extremely interesting and very general theorems concerning the possibility of transmitting messages at certain rates and with certain accuracies. Roughly, he gives a precise numerical meaning to the intuitive idea that we can trade accuracy for speed and conversely, but we cannot enter into these questions here.

Before ending this section, we should mention that Shannon's original derivation of the measure is a good bit more elegant that this one, and it is correspondingly more difficult. The main difference is the choice of domain for the measure. He began with a function defined over probability distributions, i.e., a function H having a typical value

$$H[p(1), p(2), \ldots, p(n)].$$

He did not suppose, as we did, that H can be expressed as the expected value of some function defined in terms of the probability of an individual element being selected. Rather, he gave an axiom system, which is somewhat similar to ours, for H from which he was able to derive that H _must be_ the expected value of the logarithm of the _a priori_ probabilities.

*_Problems_

1-Evaluate $-\sum_{i=1}^{n} p(i)\log_2 p(i)$ when

i. $p(i) = 1/n$, for $i = 1,2,\ldots,n$.

ii. $p(1) = 1$, $p(i) = 0$, for $i = 2,3,\ldots,n$.

(These two values can be shown to be the maximum and minimum, respectively, of the information measure when there are n alternatives.)

2-In the game "20 questions" one supposedly can isolate any "thing" in 20 binary (yes-no) questions or less. If this is so, what is the maximum possible number of "things?"

3-Suppose that A is a set of n elements with a given probability distribution $p(i)$, $i \in A$. Suppose we form a single long message by making successive independent selections from A. Let p denote the probability of this message and let N, where N is very large, denote its length. Show that $\frac{\log_2 p}{N}$ is approximately equal to $\sum_{i=1}^{n} p(i)\log_2 p(i)$.

*3.9 THE SHAPLEY VALUE OF A GAME

As background for this section you should reread section 1.9 on Legislative Schemes, particularly the last two pages. You will recall that we stated, but did not demonstrate, that for games (conflicts of interest) with n players it is possible to calculate a plausible measure of the "strength" of each coalition (= subset of the set of players). Let U denote the set of players. Then, mathematically, coalition "strength" is a function v, with domain 2^U and range the real numbers, that satisfies two conditions, namely:

i. $v(\phi) = 0$,
ii. if $A, B \subset U$ and A and B are disjoint, then
$v(A \cup B) \geq v(A) + v(B)$.

Such a function is known as the characteristic function of a game.

Most, but not all, of the present theory for games with $n > 2$ players is based entirely upon characteristic functions, and so it is really quite immaterial exactly how they arise from the original formulation of a game in extensive form (see Section 2.6). Indeed, we shall make no verbal distinction between a game and its characteristic function, and we shall speak of the game v when we actually mean the game with characteristic function v. In this section we shall be concerned with a priori measures of individual power for games in characteristic function form.

The problem we shall describe was both raised and solved by L.S. Shapely in "A value for n-person games", Contributions to the Theory of Games, II (H. W. Kuhn and A. W. Tucker, eds.), Annals of Mathematics Study 28, Princeton University Press, 1953, 307-317. An interesting and easily understood application of the Shapley value to legislative schemes is found in Shapley, L. S. and Shubik, M. "A Method for Evaluating the Distribution of Power in a Committee System," Amer. Pol. Sc. Rev., 48, 1954, 787-792.

Suppose that you are to be a player in a game (described in characteristic function form) which you have not participated in before. You cannot know exactly what will happen in the play of the game, for that depends upon decisions of other players as well as yourself. Nonetheless, it would be surprising if you did not have some opinions, based entirely upon the structure of the game as described by its characteristic function, of its a priori worth to you. For example, the legislative scheme in which a coalition is winning only if you are in it is surely worth more to you power-wise than the scheme where any coalition having a majority of the players is winning. The problem is to make such subjective evaluations both explicit and precise. From what we have said, the evaluation must depend in some manner upon the set of numbers $v(A)$, where $A \subset U$. Just what function of the characteristic function would be reasonable to select is not, on the face of it, obvious, and certainly any ad hoc definition would be questioned and countered by other suggestions. So we are driven once again to employ the axiomatic method. Following Shapley, we shall list three apparently weak conditions from which it is possible to derive the unique function which satisfies them. We will not actually carry out the proof, but we will state the result.

We start out with the idea that player i's evaluation of a game v is a real number which depends upon the characteristic function v. That is, for each player i we will have a function ϕ_i with domain the set of all possible characteristic functions and range the real numbers. The quantity $\phi_i(v)$ will be known as the value of the game v to player i.

Since the numbering of the players is arbitrary, we may always renumber them in any way we like by a permutation of the original numbering system. This will cause the characteristic function to look different even though it represents the same underlying game, but, since these are only notational differences, players who correspond under the relabeling should have the same value. So Shapley's first condition is

Axiom 1. Value shall be a property of the abstract game, i.e., if the players are permuted, then the value to player i in the original game shall be the same as the value to the permutation of player i in the permuted game.

If U is the set of all players, it is easy to show (using condition ii of a characteristic function) that no coalition has power in excess of v(U). Thus, in a sense, this is all the power available in the situation for distribution among the players. Now, although each of the players is evaluating the game for himself, his expectation must reflect in large part what the other players can rightfully expect. We would hold that if these a priori expectations totaled to more than v(U), then surely at least one of the players must be over-evaluating the worth of the game to himself. Similarly, if the sum of the values is less than v(U), then in a sense there is some under-estimation of the a priori worths. The second argument is much less convincing than the first, but let us accept it and so impose

Axiom 2. For every game v,

$$\phi_1(v) + \phi_2(v) + \ldots + \phi_n(v) = v(U).$$

Next, consider a player i who is participating in two different games with characteristic functions v and w, say. He has an evaluation for each of these games: $\phi_i(v)$ and $\phi_i(w)$. Now, if we could think of these two games as being a single game, let us call it u, then he would have an evaluation $\phi_i(u)$, but since we assume that u is but a renaming of the two given games, we should have

$$\phi_i(u) = \phi_i(v) + \phi_i(w).$$

The next thing to consider is whether we can treat the two games as a single one. Let us suppose that v is a game over the set of players R and that w is a game over the set S. While in our preceding discussion we assumed that R and S overlapped, at least to the extent of player i, we shall now be more general and suppose that they may or may not overlap. It is a trivial matter to extend both v and w to the set of all players, $R \cup S$. If A is a subset of $R \cup S$, we define

$$v(A) = v(R \cap A) \text{ and } w(A) = w(S \cap A).$$

This is to say, in the game v, a coalition A has exactly the strength given by those members of A who are actually in the game, i.e., those who are in R; the members from S who are not in R contribute nothing. Now, the two games are defined over the same set of players. Consider what may be called the sum of the two games, denoted by $u = v + w$, and defined by the condition that if A is a subset of $R \cup S$,

$$u(A) = v(A) + w(A).$$

It is easy to see that u is a characteristic function, and so it will serve as the single game representing the two given ones. Thus, the third condition imposed by Shapley is

Axiom 3. If v and w are two games and if v + w is defined as above, then

$$\phi_i(v + w) = \phi_i(v) + \phi_i(w).$$

The last axiom is not nearly so innocent as the other two. For, though v + w is a game composed from v and w, we cannot in general expect it to be played as if it were the two separate games. It will have its own structure which will determine a set of equilibrium outcomes which may be very different from those for v and for w. Therefore, one might very well argue that its a priori value should not necessarily be the sum of the values of the two component games. This strikes us as a flaw in the concept of value, but we have no alternative to suggest.

If these three axioms are accepted, then Shapely has shown that one need not -- dare not -- demand more of a value, for they are sufficient to determine ϕ_i uniquely, and, indeed, one can obtain an explicit formula for it, namely:

$$\phi_i(v) = \sum_{\substack{S \\ S \subset U}} \gamma_n(s)\,[v(S) - v(S - \{i\})],$$

where s is the number of elements in S and

$$\gamma_n(s) = (s - 1)!(n - s)!/n!\ .$$

Let us examine this formula in detail. It is a summation over all subsets of the set of players, with a typical term consisting of a coefficient -- which we shall discuss presently -- multiplying $[v(S) - v(S - \{i\})]$. If i is not a member of S, then $S - \{i\} = S$, and so the term becomes zero. Thus, the formula only depends upon those coalitions involving i. It amounts, therefore, to a weighted sum of the incremental additions made by i to all the coalitions of which he is a member.

To return to the coefficients -- the weights -- any one who has dealt at all with simple probability models will recognize them as very familiar. Suppose that we build up random coalitions by choosing a player at random from all the players, a second at random from all the remaining players, and so on. Keep track of player i and when he is added to the random coalition, calculate his incremental contribution to it. It is easy to show that the probability of his being added to $S - \{i\}$ is exactly $\gamma_n(s)$. Thus, the value of the game to him is equal to his expected incremental contribution to a coalition under the assumption that coalitions are formed at random.

*Problems

1-Suppose $n = 2$ and that

$$v(\{1\}) = -\,v(\{2\}) \text{ and } v(\{1,2\}) = 0.$$

Calculate $\phi_1(v)$ and $\phi_2(v)$.

2-Suppose $n = 3$ and that

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1.$$

Calculate $\phi_i(v)$, $i = 1,2,3$.

3.10 NON-UNIQUENESS

A consistent axiom system for a function does not always single out a unique function, and sometimes there does not seem to be any acceptable way to add another axiom to arrive at a unique function. Non-uniqueness is hinted at if all plausible attempts to prove uniqueness fail, but this is not conclusive. Basically, there are two ways to prove non-uniqueness: first, and much the most common, is to exhibit two different functions satisfying the axioms, and second, is to prove that uniqueness would lead to a contradiction.

The several solutions to a non-unique axiom system may be so different from one another that little can be said about their relation to each other. We know, for example, the whole family of logarithms (i.e., logarithms to different bases) satisfy the functional equation $f(xy) = f(x) + f(y)$, and that they are the only solutions if f is continuous. But if we do not stipulate continuity, many other discontinuous solutions exist and there is precious little that one can say about their relation to one another and to the logarithms. In such cases, one tries to avoid the resulting confusion by adding other axioms, such as continuity, so that a fairly coherent family results. By "fairly coherent" we mean that a simple description of the whole family can be found. For instance, the family of logarithms to different bases can be described by giving one member of the family -- the logarithm to a particular base -- and by noting that any other member of the family is obtained by multiplying it by a positive constant. The last statement follows from the well known property

$$\log_b x = \log_b a \log_a x.$$

Similarly, as we pointed out earlier, the several solutions to the differential equation

$$\frac{df}{dt} = -kf$$

differ from each other only by a constant.

One should not get the impression that it is always possible or desirable to introduce another axiom to restrict the functions to an easily described class. For example, in section 1.9 we stated the two conditions characterizing the characteristic functions of n-person games. It turns out that there is no simple description of the set of all characteristic functions, but we surely do not want to add more conditions. The two conditions for characteristic functions arise from considerations in game theory and they are the only ones which can be derived, so we must live with them.

One also should not get the impression, when a family of solutions can be easily described, that the only class of transformations are constants, as in the two examples above. In the next section we will present the axioms for linear utility functions, and it will be shown that two solutions u and u' of the axiom system must be related by what is known as a positive linear transformation, i.e., there are constants K and L, where $K > 0$, such that $u = Ku' + L$. And in Chapter 4, where we discuss the axiomatization of mathematical systems, we will be concerned with still another class of transformations.

Very little general advice can be given as how best to choose axioms so that the whole class of solutions can be easily described or how best to find a description of that class. Experience indicates that it is often possible to formulate problems whose solutions have a compact description and that such problems are of interest, but little of the experience in doing this has been neatly summarized, even as rules of thumb.

*3.11 THE NON-UNIQUENESS OF LINEAR UTILITY FUNCTIONS

The preferences of individuals, organizations, and industries play an important role in all behavior and, therefore, are bound to be an integral part of any behavioral science. For example, in any conflict of interest (the mathematical model being a game) each participant is confronted by decisions to be made, and, depending upon which are actually made, certain consequences result. A unique pure consequence will not necessarily arise from a given set of decisions, for there may well be probabilistic elements in the situation which together with the decisions made by the participants determine the resulting pure consequence. So, from the point of view of the participants, a probability distribution over the pure consequences is the normal result of a set of decisions, but that too can be considered to be a consequence. In any event, it is the consequences, not the decisions themselves, which matter to the participants and about which they have preferences. Of course, they will try to argue back from their preferences among the consequences to the appropriate decisions to achieve preferred outcomes, and indirectly this induces a preference structure over the decisions. This backward inference and the problem of which decision is appropriate for getting what one prefers is what game theory is about.

Preferences are relations: given two alternatives a and b, then either a is preferred to b, b to a, a is indifferent to b, or they are not comparable according to preference. But it is none too easy to work directly with relations, and certainly our theoretical powers would be vastly augmented were we able to cast them in a numerical framework, thus putting at our disposal much of ordinary mathematics. So, we are led to inquire whether there is any plausible way to assign numbers to the elements of a set A of alternatives in such a way that numerical magnitude reflects a person's preference relation $\underset{\sim}{>}$ over A. That is, if $a \in A$ and we assign $u(a)$ to a, then we would want these numbers to have the property that $a \underset{\sim}{>} b$ if and only if $u(a) \geq u(b)$. In more formal terminology, we would like to find a real-valued function u with domain A which preserves the ordering $\underset{\sim}{>}$ on A. See section 3.3 for the definition of an order preserving function.

For one important set of alternatives this is trivial to do. If a person is offered two sums of money, it seems safe to suppose that he will prefer the larger (other things being equal), and so the numerical magnitude of money serves very well. But most alternatives do not allow such a trivial assignment -- consider preferences among the several drinks offered at a party, or among automobiles, or women, etc. Actually, in these latter examples the trouble is not really in assigning numbers, for that can be easily done if the preference ordering is a weak ordering (see section 2.7), but rather that it can be done in so many different ways. Observe that if A is a

131

finite weakly ordered set and u is a real-valued order preserving function with domain A, then any strictly increasing monotonic function (see section 3.3) iterated (see section 3.2) on u is also a real-valued order preserving function with domain A. Thus, there is an infinity of real-valued order preserving functions. Furthermore, it is not difficult to show that the only property of numbers which is held fixed under such transformation is ordering; this means that we cannot use any of the ordinary properties of numbers other than ordering. In effect, then, it is pointless to replace the given weak ordering by a numerical function.

Indeed, the history of the utility concept in economics suggests that it was a good deal worse than pointless to introduce utilities in this fashion. There was so much misuse and misunderstanding that this concept of utility was pretty thoroughly discredited.

Another strand of the history of the utility idea traces back to early work, arising largely from the needs of gamblers, in probability theory. In essence, the utility problem was this: suppose you have assigned the utility u(a) to alternative a and u(b) to alternative b, then what utility does a gamble whose outcomes are either a or b have? Let us denote by $a\alpha b$ the following gamble: a chance event (such as throwing a six with a die) has probability α of occurring and probability $1 - \alpha$ of not. If it occurs you receive alternative a, if not you receive b. Thus, $b\alpha a$ means that you receive b with probability α and a with probability $1 - \alpha$. With this interpretation, $a\alpha b$ and $b(1 - \alpha)a$ mean exactly the same gamble. The question now is whether you can express the utility of the gamble, $u(a\alpha b)$, in terms of u(a), u(b), and α so that it correctly reflects preferences among gambles and pure alternatives (which are, of course, a special case of gambles).

In the traditional gambling situations, the alternatives are money, and, as we said, it is plausible to take the utility of a sum of money to be its numerical value. So, in this context, $a\alpha b$ means you get \$a with probability α and \$b with probability $1 - \alpha$. Either a or b or both may be negative numbers, which means you lose that sum. Now, if the gamble $a\alpha b$ is repeated a large number of times, one can expect an average return per gamble of $\alpha a + (1 - \alpha)b$ dollars. This appears to be a suitable index for the worth of the gamble. Or does it?

Consider the following problem, due to Bernoulli, which is known as the St. Petersburg paradox. A fair coin is tossed as many times as necessary for a head to appear. If the first head appears on the n^{th} toss, you receive 2^n dollars. Since you inevitably receive some money, the person running the game must charge a fee for each play. Question: how large does the fee have to be before you will be unwilling to pay it to play the game? According to our preceding discussion, this should depend upon your expected winnings in the game. So let us compute these. The probability that the first toss is a head is 1/2, in which case you receive 2 dollars; the probability of a tail on the first toss and a head on the second is $(1/2)(1/2) = 1/4$, in which case you receive $2^2 = 4$ dollars; the probability of tails on the first two tosses and a head on the third is $(1/2)(1/2)(1/2) = 1/8$, in which case you receive $2^3 = 8$ dollars; etc. Since each of these possible outcomes is independent of the others, we compute the expected winnings as

$$2(1/2) + 4(1/4) + 8(1/8) + \ldots + 2^n(1/2^n) + \ldots = 1 + 1 + 1 + \ldots\ldots,$$

which sums to no finite amount. In other words, you should be willing to pay any finite amount, however large, to participate once in this game. But this is silly.

There appear to be two possible flaws in the argument that leads to the St. Petersburg paradox. First, we assumed that the worth, or utility, of money to a person is measured by its numerical value. There is considerable evidence, including subjective considerations, to suggest that this is false. Second, we employed an argument based on many replications of the gamble to justify evaluating it in terms of its expected monetary return, and then we used this same evaluation when the gamble occurs only once. In other words, while the long run argument probably makes sense for a gambling house, it may not for the individual gambler playing only a few times. Another indication that monetary expected values do not represent the subjective worths of gambles is based on the following observation. Consider gambles of the following form: you win a fixed amount x if a fair coin comes up heads and lose the same amount x if it comes up tails. All such gambles have the same expected return, namely: 0. But would you be indifferent between one where $x = \$0.05$ and one where $x = \$1,000$? Obviously not.

So, we may want to drop either the assumption that the utility of money is equal to its numerical value or the assumption that the utility of a gamble is given by the expected value of the utilities of its components or both. Of the two, we are much more willing to drop the first than the second, especially since it only applies to money anyhow. For non-monetary outcomes we have the task of assigning numbers and so we might just as well extend this problem to include monetary outcomes. But once we admit that the utility of money may be different from its numerical values, then the expected value of utility (not money!) assumption may hold. At least we do not have any evidence to hand which shows that it doesn't. The modern theory of utility, which originated with von Neumann and Morgenstern in the second edition of their famous book The Theory of Games and Economic Behavior, Princeton: Princeton University Press (1947), describes the conditions on the preference ordering such that one can work with expected utilities. For a general survey of more recent work in this and related topics, see Edwards, W., "The Theory of Decision Making," Psychol. Bull., 51, 1954, 380-417 and Luce, R.D. and Raiffa, H., Games and Decisions Wiley, 1957, Chapter 2.

The reason for all this fuss about expected utilities is largely mathematical, for without this property such theories as those of games and statistical decisions would be virtually impossible.

Intuitively you can easily see the power of the assumption. One need not know the utility function for each of the infinity of gambles possible with a finite set of alternatives, rather it is sufficient to know them for the finite set and to compute them, using expected utilities, for any gamble. It permits an extremely economical summary of a person's preferences over all gambles.

So we have the following problem. Let a finite set A be given and let G be the set of all possible gambles formed from elements of A. Let G be weakly ordered by $\gtrsim$. To find a real-valued function u with domain G satisfying

Axiom 1. (order preserving) if $a,b \in G$, $a \gtrsim b$ if and only if

$$u(a) \geq u(b).$$

Axiom 2. if $a,b \in G$ and α is any real number such that $0 \leq \alpha \leq 1$, then

$$u(a\alpha b) = \alpha u(a) + (1 - \alpha)u(b).$$

Any function satisfying these conditions is known as a linear utility function of the weakly ordered set G. (The word "linear" refers, in this context, to the second axiom.)

The first observation we make is that this axiomatization is not consistent unless the weak ordering satisfies certain restrictive properties. For example, if $a \gtrsim b \gtrsim c$ it is necessary that there be a number α, $0 \leq \alpha \leq 1$, such that $b \sim a\alpha c$. To show this, suppose u satisfies the axioms. Then by axiom 1, $u(a) \geq u(b) \geq u(c)$. An elementary property of numbers assures us that there exists a number α, $0 \leq \alpha \leq 1$, such that $u(b) = \alpha u(a) + (1 - \alpha)u(b)$. Thus, according to axiom 2, $u(b) = u(a\alpha c)$, so by axiom 1, $b \sim a\alpha c$.

This means, in effect, that preferences must possess a certain continuity if they are to be represented by linear utility functions. If we think of α as a variable quantity, b is preferred to a when $\alpha = 0$ and as it is increased until a point is reached where they are indifferent. After that, any increase in α causes the gamble to be preferred to b. Except for certain discrimination difficulties which people always seem to exhibit, this seems like a plausible way for preferences to behave. But there may be exceptions, as is suggested by letting

a = five cents
b = one cent
c = instant death.

One can derive other necessary requirements on the weak ordering if the axioms are to be consistent, i.e., if a linear utility function is to exist. Furthermore, and this is the important part of utility theory, von Neumann and Morgenstern took one such set of conditions -- each of which has a certain intuitive plausibility for preferences -- and showed that whenever these are met there must be a linear utility function. We shall not develop this theory in this section; our aims are more modest.

We shall suppose that we have a case where a linear utility function exists and then inquire into its uniqueness properties. First, it is easy to see that it is not unique, for if u satisfies the axioms and K is a constant > 0, then so does Ku. Second, since there are several linear utility functions, we would like to know how they are related to one another. Our claim is this:

Theorem. If u and u' are two linear utility functions, then there exist constants K and L, $K > 0$, such that

$$u = Ku' + L;$$

and any such transformation, which is known as a positive linear transformation, of a linear utility function is also a linear utility function.

Proof. The second half of the assertion is easily verified and it is left as a problem.

The first part is a little more subtle. Choose any $a, b \in G$ such that $a > b$. Define K and L to be solutions to the following simultaneous algebraic equations:

$$u(a) = Ku'(a) + L$$
$$u(b) = Ku'(b) + L;$$

i.e.,

$$K = \frac{u(a) - u(b)}{u'(a) - u'(b)}$$

and

$$L = \frac{u'(a)u(b) - u(a)u'(b)}{u'(a) - u'(b)}$$

Since $a > b$, axiom 1 implies that $u(a) > u(b)$ and $u'(a) > u'(b)$, so both constants are well defined and $K > 0$.

Now, consider the function

$$u'' = Ku' + L.$$

We claim that $u'' = u$, i.e., for every $c \in G$, $u''(c) = u(c)$. Since $\gtrsim$ is a weak ordering, exactly one of the following three cases holds for each $c \in G$:

i. $c > a$,

ii. $a \gtrsim c \gtrsim b$,

iii. $b > c$.

Each one requires roughly the same treatment; we will carry out case ii here, and i is assigned as a problem and iii is almost identical to i. In case ii, we know from the necessary condition we derived for $\gtrsim$ that there exists a number α, $0 \leq \alpha \leq 1$, such that $c \sim a\alpha b$. According to axiom 1,

$$u''(c) = u''(a\alpha b) = Ku'(a\alpha b) + L.$$

Since u' satisfies the axioms, we use the second one and carry out some simple algebra:

$$\begin{aligned} u''(c) &= Ku'(a\alpha b) + L \\ &= K[\alpha u'(a) + (1 - \alpha)\, u'(b)] + L \\ &= \alpha[Ku'(a) + L] + (1 - \alpha)[Ku'(b) + L]. \end{aligned}$$

Recalling the equations which were used to define K and L,

$$\begin{aligned} u''(c) &= \alpha[Ku'(a) + L] + (1 - \alpha)[Ku'(b) + L] \\ &= \alpha u(a) + (1 - \alpha)u(b). \end{aligned}$$

Finally, we use the fact that u satisfies axiom 2 to obtain

$$\begin{aligned} u''(c) &= \alpha u(a) + (1 - \alpha)u(b) \\ &= u(a\alpha b) \\ &= u(c). \end{aligned}$$

So we have completed the proof that u and u' must be related by a positive linear transformation. Thus, in any particular problem it is sufficient to describe one of the linear utility functions in detail, i.e., to give its values on the underlying set A, and to remark that all others are related to it by means of a positive linear transformation.

Another way to say this is that linear utility functions are uniquely determined up to their unit and zero, i.e., it is completely arbitrary which element of G we take to have zero utility and which pair of elements we take to be one unit apart in utility. This means that it is a measurement like ordinary (Fahrenheit or Centigrade) temperature scales, not like length or mass where the zero is uniquely determined.

*Problems

1-Suppose u is a real-valued order preserving function with a quasi-ordered domain. Show that the quasi-order must, in fact, be a weak order.

2-If $K > 0$, show that $Ku + L$ must be a linear utility function if u is.

3-Carry out the proof that $u''(c) = u(c)$ for the case $c >' a$.

4-From the axioms for a linear utility function, show that $\gtrsim$ must have the property that

$$a\alpha b \gtrsim a\beta b \text{ if and only if } \alpha \geq \beta.$$

Interpret in words what this means. Is it a reasonable condition?

CHAPTER IV

AXIOMATIZATION OF MATHEMATICAL SYSTEMS

4.1 INTRODUCTION

From the mathematician's point of view, our failure to emphasize any axiomatizations other than of functions has been, to say the least, peculiar. He would feel, and rightly for pure mathematics, that our present topic is much the most important aspect, that our long chapter 3 should have been little more than a footnote to this chapter, and, when he had completed this chapter, he would feel we had done a very incomplete job. Without disputing such objections for pure mathematics, we feel -- and at present this is little more than a conjecture -- that our emphasis is reasonable for those who will be concerned with applications of mathematics to behavioral problems. Nonetheless, because the history is short and also because tradition is always an uncertain guide to the future, we would be unwise not to suggest the more prevelant uses of the axiomatic method in mathematics and to indicate some of the systems which have proved important.

By a mathematical system we have in mind something fairly complex, usually with several interrelated operations, which is studied as a whole entity. Examples are: geometry, the real or complex number system, the algebra of matrices, the theory of sets, etc. Other examples, derivative from these, will appear later. In the course of studying such systems, studies that initially are very fumbling and tentative, certain concepts and operations gradually loom as more important than others. They seem to be more fundamental to the system in the sense that they are widely used and are often crucial in the proofs of important theorems. Sometimes these are the ideas which have arisen early in the study and which seem intuitively natural; in other cases they seem to be much more sophisticated concepts which have required a long time for their development. For example, in the number system the operations of multiplication and addition, which arose very early, are still considered basic to the system and are the source of ideas for a great deal of modern algebra. But equally well, the idea of a topology -- the "near to" structure mentioned earlier -- appears today to be a crucial feature of the number system, and it has led to the extremely fruitful study of topological spaces which pervade much of modern mathematics. This idea was much slower developing, and it was really only adequately formulated within the last fifty years.

As certain concepts and operations begin to stand out as crucial to the system being studied, one is tempted to isolate them totally from the original system and to study them in their own right. This is the central idea lying behind the axiomatization of mathematical systems. Example: if we think of real numbers, they have a lot of properties: a notion of multiplication, of addition, of less than, of nearness, etc. We could single out just one of these for isolated study, ignoring its relations to the others. For instance, suppose we select multiplication. The minute we do this, we begin to realize that we have concepts of multiplication in other mathematical systems, such as matrix algebra, and so one is lead to see what common properties multiplication may have in these several systems. Such a study finally leads to the very rich theory of groups.

By isolating a portion of a system we mean this: One or more operations or concepts of a known system are selected and some of their properties, i.e., theorems in the original system, are taken as axioms to characterize these "undefined" operations or concepts. The choice of which properties to use is, just as with functions, a fairly subtle business, requiring judgment and experience.

Once an axiomatization is given, then the mathematical problem is to introduce definitions -- often motivated by corresponding concepts in the original mathematical system which suggested the axiomatization -- and to prove theorems about these definitions, i.e., true assertions which follow from the axioms and the rules of logical inference. Again, many of the central theorems, at least at the start, will be suggested by properties found in the original mathematical system; however, as the axiom system is intensively studied, there will usually result ideas and theorems either not noticed or not particularly significant in the original system.

You may ask: what is the point of all this? Our answer must be that such methods have, historically, enriched mathematics and science. For one thing, isolating a part of a complex system such as the real numbers permits us to see which classes of theorems rest upon which basic facts. Second, some of the systems which have been isolated have been found to recur over and over in widely divergent parts of mathematics, and so their independent study has meant that an elaborate set of theorems are ready for application in any context where operations and concepts satisfying the axioms appear. The abstract notion of multiplication as formulated in group theory is a case in point. Third, isolating significant subsystems of one or more basic mathematical systems permits us to see how they might be recombined in a variety of ways. Some of these new constructs have in the past proved extremely fruitful in extending our understanding of one part or another of mathematics and in creating new mathematics which, sometimes, is suited to particular applications. The recent history of mathematics includes the intensive study of systems, the abstracting of portions as axiomatic systems, the application of these results in other parts of mathematics and the recombining of different subsystems to form new mathematical systems, the intensive study of these, further abstraction, etc. This continual and complicated interplay and refertilization arising from abstracting and recombining has proved very stimulating to mathematics.

Although we have emphasized self stimulation in mathematics, we do not want to play down the possibly more important stimulation furnished by the application of mathematics to science. It is here, primarily, that the original mathematical systems are developed and the central theorems, which often are suggested by the properties of the physical systems abstracted, are proved. It is anticipated that the attempted applications of mathematics to behavioral problems will, in the future, prove to be the source of many new and rich ideas for mathematics.

Some, though not all, axiomatizations of mathematical systems are a good deal like the material discussed in chapter 3 in that they involve functions. Even so, there are two ways in which they generally differ, although neither of these is strictly necessary. First, in very many cases there is more than one function, and the several functions are intertwined in some manner. Example: if we were to abstract from the theory of sets, we would have both a function representing union and one representing intersection and these would have to be interrelated in just the way union and intersection are in set theory. Second, when an axiomatization includes functions, it is usual for the range and the domain of the function to be extremely closely related. Example: suppose multiplication of numbers is to be abstracted. In that case one assigns to every pair of numbers, a and b, a third number ab called the product of a and b. This suggests that in the abstract formulation there must be a function with some set R as its range and $R \times R$ as its domain. Or, if we try to abstract set theory, one operation which must be taken into account is complementation, which assigns to every subset of U another subset of U, namely, its complement. Thus, in the abstraction we would have a function with both domain and range some set A, where A plays the role of 2^U. In the usual terminology, many of the operations which concern us are "closed." We take one or more elements from a given set and the operation leads us to another element in the same set. This sort of closure is reflected by having the domain of the function which represents the operation closely related to its range.

But one should not get the idea that all of the systems studied axiomatically are based on functions. Actually, we have already seen several cases which were not, although at the time we did not mention that we were using the axiomatic method. Our definitions of different classes of relations -- reflexive, symmetric, etc. -- were really simple axiom systems. Similarly, our definition of an algebra of sets could be treated as an axiom system.

4.2 SOME TERMINOLOGY

As in the axiomatization of functions, one has problems of existence, uniqueness, etc. Actually, slightly different questions must be phrased and some of the emphasis is, for good reason, different.

Consistency. The first question here, as with functions, is whether the axioms as a group are consistent, i.e., whether something exists which satisfies them. It is usual to call any mathematical structure which satisfies all the axioms of a given system an interpretation of the system. For the most part, this problem is either extremely simple or extremely difficult, depending upon how you look at it. Since the axioms in general arise from some special mathematical structure, such as the real numbers or set theory, which has already been investigated and is generally accepted to exist, they are trivially consistent. This is the simple way of looking at it. But one can question how we really know that the number system exists. Are we not living in a fool's paradise by supposing that we cannot prove contradictory results within that system? While this is a fruitful skepticism from a pure mathematics point of view, in applied work one usually takes a far more pragmatic approach and assumes that the well known systems do in fact exist and that they will serve as demonstrations of the consistency of an axiom system.

Completeness. The direct analogy of uniqueness for axiom systems is completeness, meaning, roughly, that there is at most one possible interpretation of the system. By and large, this concept is not of the same practical importance for systems as it was for functions, because most of the axiomatizations are not complete. Indeed, much fundamental research on the completeness of systems, stemming in large part from work of Gödel, has shown that it is very rare to find a set of axioms which are both consistent and complete. Such work has been extremely important in understanding the foundations of mathematics and in some respects it has been profoundly disturbing, but for applications it need bother us but little. Of much greater relevance to us is the next notion.

Categoricalness. An axioms system will be called categorical if, roughly, any two interpretations of the system, while not identical, are formally the same in the sense that one can be "superimposed" on the other in such a way that they look alike. That is to say, elements of one can be identified with elements of the other and operations in one with operations in the other, so that corresponding operations take corresponding elements into corresponding elements. We will go into what we mean by this much more fully in the next section, for it is an important notion.

In practice, this amounts to having uniqueness, for it means that if we have investigated one interpretation of the system fully, then we know how all others must look, for we can set up an identification of elements and operations so that they are formally the same.

When an axiom system is not categorical, then problems arise which are similar to non-unique axiomatizations of functions. Can we establish how one interpretation maps into another? If not, can we classify the several interpretations into some reasonable taxonomy?

Independence. One axiom of a system is said to be independent of the others in the system if it is not a logical consequence of them. One shows this is so by finding an interpretation which meets all the remaining axioms but which fails to meet the one under consideration. It is clear that if such an interpretation can exist, then it is impossible to derive the one axiom from the others. If this can be done for each of the axioms in a system, then the system is said to be independent.

This notion is not really terribly important for most purposes. To be sure, it is nice to get rid of obvious redundancies in an axiom system, but often the less obvious ones are left in for either psychological or pedagogical reasons. A redundant axiom system is often much more intuitive and easier to recall than one that is independent. Of course, it can be an intriguing mathematical game to be certain that a particular set of axioms is independent and, if not, to devise one that is, but only rarely does this result in a valuable contribution to the understanding of the system. Some judgment is generally necessary.

Weakness and Strength. Allied to the concept of independence is the scientifically more important question of which of two axiom systems for the same concepts is the weaker. Such problems arise from the fact that there is never a unique way to axiomatize any concept. If we are abstracting from a given mathematical system, we choose certain properties of the original system as axioms and from these derive other properties as theorems. It is quite arbitrary which of the properties we choose to take as axioms and which we try to derive, though in general there is not really complete freedom in this choice. The axioms should be, in some sense, simple, immediate of comprehension, and appealing to the intuition. But even within these vague criteria, there is a good deal of freedom in their choice.

Now suppose A and B are two axiom systems for the same undefined concepts. We say that A is stronger than B (equally, B is weaker than

A) if it is possible to derive all of the axioms of B from those of A, but the converse is not possible. If we can derive all we want or need from a weaker axiom system we shall always prefer it to the stronger one. The reason for this is clear: we invariably want to assume as little as possible to get the results we need, and so of two axiom systems which give us these results we prefer the weaker one. This does not deny that from the stronger system we can derive theorems which we cannot prove for the weaker one, but rather that, for whatever reason, we are not interested in these extra results.

It should be mentioned that it is not always easy to apply the above principles. A particular theorem which we need may appear to require all the axioms of a particular system A, and, therefore, appear not to be provable within a weaker axiom system B; yet it is well known that appearances can be deceiving. An ingenious mathematician may be able to derive the theorem from B, even though the original proof seemed to rest on everything assumed in A. Often this can be a very valuable contribution as, for example, when the axioms of A seem too strong to be tenable in some empirical context, but those of B are acceptable.

4.3 CATEGORICALNESS AND ISOMORPHISM

As we pointed out above, it is fairly rare for an axiom system to be complete, i.e., to have a unique interpretation, but it is much more common for one to be categorical, i.e., for the interpretations to be formally the same. In this section we wish to make clear, without being completely precise about it, what we mean by two systems being "formally the same." The word which is used for this notion is isomorphism. What we will do is define isomorphism of two very simple classes of systems and then suggest how it must be defined more generally.

The simplest case is simply that of sets having no structure of any sort upon them. In that case a special word instead of isomorphism is used. Two sets A and B are said to be in one-to-one correspondence (usually written 1:1 correspondence) if there exists a function f from A onto B which has an inverse. Let us see what this means. First, since "function" means "single-valued function" there is just one element of B associated to any one element of A, and, since the function is onto and has an inverse, there is associated to each element of B just one of A. In other words, there is a one-to-one pairing of the elements of A and B.

It is easy to see that if A and B are finite sets, then they can be placed in 1:1 correspondence if and only if they have the same number of elements. But be careful about carrying over notions of the meaning of 1:1 correspondence from finite sets to infinite sets. If A is finite, there clearly cannot be a proper subset B which is in 1:1 correspondence with A. But if A is infinite, this is possible (indeed, it is one way to define what we mean by infinite). Consider, for example, A = set of integers and B = set of even integers. Clearly, B is a proper subset of A since all the odd integers are not included in B. But we claim that there is a 1:1 correspondence between A and B, namely $f: A \to B$, where

$$\text{if } a \in A,\ f(a) = 2a.$$

Since a is an integer, 2a is an even integer. The mapping is onto since if 2a is an even integer, a is an integer. And f has an inverse, namely, $f^{-1}(b) = b/2$, $b \in B$.

Any infinite set which can be put into 1:1 correspondence with the integers is said to be a countable (or denumerable) set; otherwise it is said to be non-countable. By what we have just seen, the even integers are countable. Equally well, so are the odd ones. Less obvious is the fact that the set of all fractions (numbers of the form a/b, where a and b are integers) is also countable. If that is so, one might be tempted to suppose that all infinite sets are countable, but we can show that this is not the case. Consider the set of all real numbers lying between 0 and 1. We show that this set is not countable. To do this, we suppose that it is in fact countable, i.e., there is a first (the one mapped into the integer 1) which we denote by a_1, a second (the one mapped into the integer 2) which we denote by a_2, etc. It is, of course, well known that any real number can be expressed as an infinite decimile expansion of integers from 0 through 9. Let us denote the i^{th} integer in the expansion of a_n by a_{ni}, i.e., we have the array

$$\begin{array}{l} a_1 = 0.a_{11}\ a_{12}\ a_{13}\ldots\ldots \\ a_2 = 0.a_{21}\ a_{22}\ a_{23}\ldots\ldots \\ a_3 = 0.a_{31}\ a_{32}\ a_{33}\ldots\ldots \\ \cdot \\ \cdot \\ \cdot \\ a_n = 0.a_{n1}\ a_{n2}\ a_{n3}\ldots\ldots a_{nn}\ldots\ldots \\ \cdot \\ \cdot \\ \cdot \end{array}$$

Now, let us consider the real number

$$b = 0.b_1 b_2 b_3 \ldots\ldots ,$$

where

$$b_i = \begin{array}{ll} a_{ii} + 1, & \text{if } a_{ii} \neq 9 \\ 0 & \text{, if } a_{ii} = 9 \end{array} .$$

We claim that our counting has ignored this number. If not, then it is some number in our list, say the n^{th}. Now consider the decimile expansions of a_n and b. By choice the n^{th} integer in the former is a_{nn} and in the latter $b_n \neq a_{nn}$. Thus, they are not the same number,

149

so b was omitted from the counting. But this is contrary to choice, so such a counting cannot be possible.

So, for sets without any assumed structure, 1:1 correspondence is what we mean by isomorphism. Whenever there is some structure under consideration, the idea becomes a little more complex. Possibly the simplest case is sets having a single relation defined on them.

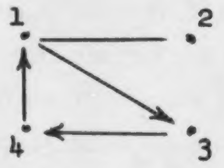
Suppose A and B are sets with relations R and S, respectively. We say the system (A,R) is isomorphic to the system (B,S) provided we can find a 1:1 correspondence f between A and B such that both f and f^{-1} are order preserving, i.e.,

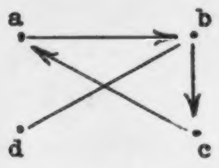
$$\text{if } a,b \in A, \text{ then } aRb \text{ implies } f(a)Sf(b),$$

and

$$\text{if } x,y \in B, \text{ then } xSy \text{ implies } f^{-1}(x)Rf^{-1}(y).$$

Graphically, this is particularly easy to see. Consider the two relations





We claim that these are isomorphic. This is easily checked once you make the 1:1 correspondence of the points:

1-b
2-d
3-c
4-a

Another case where isomorphism is simple to define is among sets having a multiplication structure. As we pointed out before, multiplication in a set A amounts to a function from $A \times A$ into A. It is usual, however, not to use functional notation, but rather some symbol such as $a \circ b$ or $a * b$ to stand for the element which is the "product of a and b." Suppose $(A, \circ)$ and $(B, *)$ are two multiplicative systems. We say that they are isomorphic provided that we can find a 1:1 correspondence f between A and B such that f and f^{-1} both preserve the multiplication, i.e.,

$$\text{if } a,b \in A,\ f(a \circ b) = f(a) * f(b),$$

and

$$\text{if } x,y \in B,\ f^{-1}(x * y) = f^{-1}(x) \circ f^{-1}(y).$$

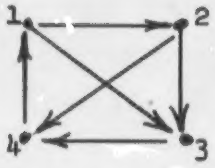
The generalization to more complicated mathematical structures is probably clear. We begin with two sets, each having a series of operations, functions, relations, and the like. A 1:1 correspondence is established between the operations, etc., and another between the elements of A and B. If the element-wise correspondence has the property that both it and its inverse preserve the effects of corresponding operations, then we say it is an isomorphic mapping of the one system onto the other. If such a mapping exists, then the two systems are said to be isomorphic. We will not try to make isomorphism any more precise than that.

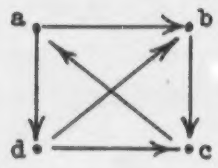
As we said earlier, most axiom systems are not complete, but many are categorical in the sense that any two interpretations are isomorphic.

Problems

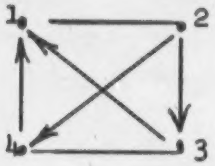
1-Establish whether or not the following pairs of relations are isomorphic.

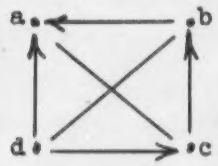
i.





ii.





2-Let M denote a class of sets, each set having a relation defined over it. Consider isomorphism of pairs of sets and their relations as a relation over M. Show that it is an equivalence relation.

3-Show that the positive real numbers under multiplication are isomorphic to the set of all real numbers under addition.

4.4 BOOLEAN ALGEBRAS

Our first, and primary, example of an axiomatic system arises from set theory. The interpretations of the system that we shall present are known as Boolean algebras, the name honoring George Boole who laid the foundations of set theory. In one way it is a poor example of axiomatization, for it is somewhat unrepresentative. Rather than select some portion of set theory to serve as a guide for the axiom system, we shall use the entire structure: axioms will be given for "undefined" operations analogous to all of those of set theory -- to union, intersection, complementation, and inclusion. It is much more usual to attempt to abstract some limited, but crucial, portion of the original system; we shall see two examples of this using the real number system as the source of ideas. Actually, there are such abstractions from set theory. For example, if one abstracts the inclusion relation and the fact that the union of A and B is the smallest set which includes both A and B and the intersection of A and B is the largest set included both in A and B, then one has the axioms for what are known as lattices. But because Boolean algebras are quite important and we already have some background in set theory, we shall use them -- even though they are not fully representative -- as our example of an axiomatization.

Let us suppose that we impose on a set B four undefined operations which correspond to the four major operations in set theory, and we specify that they must satisfy properties which correspond to some of the theorems which can be derived in set theory. But there are an infinity of true theorems in set theory, so which shall we select? This problem always confronts one in an axiomatization, and a wise choice among these possible axioms is always a major intellectual feat. Often when an abstraction is first being formulated and investigated, several different axiomatizations will be put forward. Sometimes they all persist in the literature, but more often experience with the several formulations leads to a decision as to which is preferable. In the present context, we don't really have to enter into the pros and cons of different axioms, for there is widespread -- indeed, total -- agreement. This isn't to say that there are not a number of different systems for Boolean algebras available and used, but only that they are all equivalent to each other. The set of theorems which we shall use as axioms are those listed in section 1.6. It is recommended that you reread that section now.

So, to be more precise, we assume as given: a set B which has at least two elements ϕ and u -- these will play special roles corresponding to the null and universal sets -- , a relation R on B, and three functions

$$F_1: B \times B \to B$$
$$F_2: B \times B \to B$$
$$F_3: B \to B$$

Notational convention: Since the relation and the three functions are

153

to play roles paralleling the usual set operations, it is convenient to use a parallel notation. In actual practice, exactly the same symbols are employed; however, here we shall add a star to each of the usual symbols in order to emphasize the distinction between these operations which are to be axiomatized and the familiar set theoretical operations. Thus, if $a,b \in B$, we write

$$
\begin{aligned}
&a \subset *b && \text{for } aRb,\\
&a \cup *b && \text{for } F_1(a,b),\\
&a \cap *b && \text{for } F_2(a,b),\\
&\bar{a}^{*} && \text{for } F_3(a)
\end{aligned}
$$

It is extremely important to understand that these starred operations have nothing to do with inclusion, union, intersection, and complementation among the subsets of B. They stand for relations and functions defined in terms of the elements of B, not in terms of its subsets. In the axiomatization, the elements of B will play the role of subsets in set theory. In other words, if U is a universal set, B corresponds to 2^U, not to U.

A system $(B, \cup *, \cap *, \subset *, -*)$, where the ranges and domains of the operations are those given above, is called a *Boolean algebra* provided that the following axioms are satisfied for every $a,b,c \in B$:

Axiom 1. $a \subset *a$.

Axiom 2. if $a \subset *b$ and $b \subset *c$, then $a \subset *c$.

Axiom 3. $\phi \subset *a$.

Axiom 3'. $a \subset *u$.

Axiom 4. $a \cup *a = a$.

Axiom 4'. $a \cap *a = a$.

Axiom 5. $a \cup *b = b \cup *a$.

Axiom 5'. $a \cap *b = b \cap *a$.

Axiom 6. $a \cup *(b \cup *c) = (a \cup *b) \cup *c$.

Axiom 6'. $a \cap *(b \cap *c) = (a \cap *b) \cap *c$.

Axiom 7. $a \cup *(b \cap *c) = (a \cup *b) \cap *(a \cup *c)$.

Axiom 7'. $a \cap *(b \cup *c) = (a \cap *b) \cup *(a \cap *c)$.

Axiom 8. $\phi \cap *a = \phi$.

Axiom 8'. $u \cup *a = u$.

Axiom 9. $\phi \cup {}^{*}a = a$. Axiom 9'. $u \cap {}^{*}a = a$.

Axiom 10. $a \cup {}^{*}\overline{a}^{*} = u$. Axiom 10'. $a \cap {}^{*}\overline{a}^{*} = \phi$.

Axiom 11. $\overline{a \cup {}^{*}b}^{*} = \overline{a}^{*} \cap {}^{*}\overline{b}^{*}$. Axiom 11'. $\overline{a \cap {}^{*}b}^{*} = \overline{a}^{*} \cup {}^{*}\,\overline{b}^{*}$.

Axiom 12. $\overline{\overline{a}^{*}}^{*} = a$.

Axiom 13. Each of the following implies the other two:

$a \subset {}^{*}b,\ a \cap {}^{*}b = a,\ a \cup {}^{*}b = b$.

The first problem we must consider is whether this axiom system is consistent. It certainly is for some sets B, for all we need do is choose a set U and let $B = 2^{U}$ and the resulting algebra of subsets is certainly an interpretation of the system. A theorem which we shall state below shows that the axiom system is only consistent for certain sets B.

Second, why did we choose this particular set of theorems of set theory to use as axioms? The answer is reasonably clear if you recall the statement we made when discussing the corresponding theorems in set theory to the effect that it is possible from these theorems alone to prove any other theorem which can be phrased in set theory concepts. It is never necessary to resort to arguments involving elements of subsets. Thus, in any Boolean algebra it is always possible to derive from the axioms any theorem whose corresponding statement is true for sets. But if that is so, it must mean that any interpretation of a Boolean algebra cannot really be very different from the algebra generated by the subsets of a given set. Among other things, we might conjecture that all interpretations in which the underlying sets B are in 1:1 correspondence are isomorphic. This, and a bit more, is true, but our way of arriving at it has hardly been honest. It has been built upon our statement in Chapter 1 that any other true theorem for sets could be derived directly from those which we listed, but we did not prove this statement.

It actually follows from the central representation theorem for

Boolean algebras to which we have been leading up. The theorem is this: Any interpretation of a Boolean algebra is isomorphic to an algebra of sets (see section 1.8), which is, of course, also an interpretation. This says not only that any two interpretations which are in 1:1 correspondence are isomorphic, but also that there are no interpretations which are not isomorphic to an algebra of sets. Thus, whenever we have to think about a Boolean algebra, we will not be misled by thinking of an algebra of subsets of a given set with the operations of union, intersection, inclusion, and complementation.

But if this is so, has there really been any point to the axiomatization? Won't all interpretations be so immediately parallel to set theory itself that the axiomatization is superfluous? Two rather distinct examples suggest that this is not so. The first we shall sketch briefly now, and you will study it and related topics much more fully later; the second will be presented in the next section.

Consider elementary logic. It begins with a set of propositions, i.e., statements which can be either true or false. Examples: "a red automobile must be a fire engine," "Mt. Everest exceeds 10,000 feet in height," "a tree is a tree or a house," etc. Of these, the first is empirically false, the second empirically true, and the third tautologically true. Four basic logical connectives can be identified which allow us to form new propositions from old: If p and q denote propositions, then we can form the propositions

p or q, p and q, not p, and p implies q.

It is generally assumed that these connectives satisfy certain properties. These lie so deep in our early training, and are so closely integrated with experience, that we sometimes forget that they are unproved assumptions -- or axioms of the calculus of propositions. For example: not(not p) = p, p or q = q or p, etc.

It turns out that if we make the following identifications

"or" for $\cup^*$
"and" for $\cap^*$
"not" for $-^*$
"implies" for $\subset^*$,

then each of the axioms for a Boolean algebra becomes one of the usual assumptions of logic. In other words, the mathematical structure lying behind our ordinary propositional logic is the same as that for sets; it is a Boolean algebra. This is not unreasonable when you remember that $A \cup B$ is the set of elements either in A _or_ in B, $A \cap B$ the set of elements in A _and_ in B, etc.

One final point should be made. The axioms we have presented are far from being independent; this is suggested by axiom 13 which says that we could have defined some of the operations in terms of others. In the literature one can find a number of different independent sets of axioms for Boolean algebra, and there is one which is based upon a single undefined operation (known as the Sheffer stroke). We have chosen a non-independent set for our discussion because in this form they are particularly simple and intuitive; the axioms of the independent sets tend to be somewhat more obscure. For a much more detailed discussion of these points see Birkhoff's _Lattice Theory_ and for a less comprehensive, but simpler, discussion see Birkhoff and MacLane's _A Survey of Modern Algebra_.

Problems

1-Show that every Boolean algebra has a Boolean subalgebra of just two elements.

2-Define formally what it means for two interpretations of a Boolean algebra to be isomorphic.

*3-Let U be a set. Form the n by n matrices having as entries, not numbers as in ordinary matrix theory, but subsets of U. Thus if A_{ij} denotes the entry in the ith row and jth column of the matrix A, $A_{ij} \subset U$. Can you see plausible ways to define the Boolean operations $A \cup {}^*B$, $A \cap {}^*B$, $A \subset {}^*B$, and $\overline{A}^*$, where A and B are such matrices, so that the set of all such n by n Boolean matrices form a Boolean algebra? By analogy to ordinary matrix multiplication, we can define Boolean matrix multiplication as follows: The ij entry of the product of A and B, written AB, is the subset

$$(A_{i1} \cap B_{1j}) \cup (A_{i2} \cap B_{2j}) \cup \ldots . \cup (A_{in} \cap B_{nj}) = \bigcup_{k=1}^{n} (A_{ik} \cap B_{kj}).$$

Suppose we think of the rows and columns of the matrix as representing people and of U as a set of information. What is the analogue to the communication interpretation of ordinary matrix multiplication given in section 2.5.

4.5 SWITCHING CIRCUITS

A switch, as we shall use the term, is any device which is always in one of two possible states, which can be called "on" and "off." An electric knife switch of the type shown in Fig. 21 is a good example. Such switches are "wired" into circuits to form more complicated switches of the same general sort, i.e., under some conditions they are on, under the remaining they are off. If a switch is in the "on" position, current from the input will flow through a wire attached to the "on" terminal, but not through a wire attached to the "off" terminal. Each distinct switch in a circuit will be given a name such as p or q. In some circuits there will be two physically distinct switches which are "ganged" together so that they are on together and off together. In such cases, the same symbol will be used for these switches, for they are functionally the same.

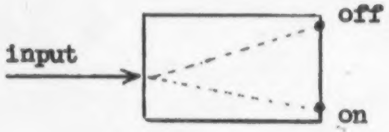


Fig. 21

We propose to show that the algebra of such circuits is a very simple Boolean algebra and that this is useful to know. Suppose it is a Boolean algebra, then since a switch has only two states it is presumable the two element algebra $\{\phi, u\}$ which is relevant. Let us identify the element ϕ with the "off" terminal of a switch and u with the "on" terminal. Now if p is a switch, we say that $p = \phi$ if the knife is in the "off" position and $p = u$ if the knife is in the "on" position. Now, consider the switch $\bar{p}$:

$$\text{if } p = \phi \text{ , then } \bar{p} = \bar{\phi} = u$$
$$\text{if } p = u \text{ , then } \bar{p} = \bar{u} = \phi.$$

So $\bar{p}$ is the switch which is on when p is off and off when p is on. Physically, $\bar{p}$ can be obtained from the switch p simply by interchanging the connections to the output terminals, as in Fig. 22.

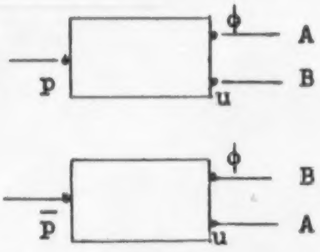


Fig. 22

Now, if p and q are two switches, then $p \cup q$ is a switch with the following properties:

$$p \cup q = \phi \text{ if } p = \phi \text{ and } q = \phi,$$
$$= u \text{ otherwise.}$$

This amounts to wiring p and q in parallel, as shown in Fig. 23.

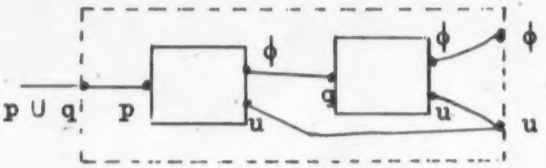


Fig. 23

Similarly, $p \cap q$ is a switch with the properties

$$p \cap q = u \text{ if } p = u \text{ and } q = u$$
$$= \phi \text{ otherwise.}$$

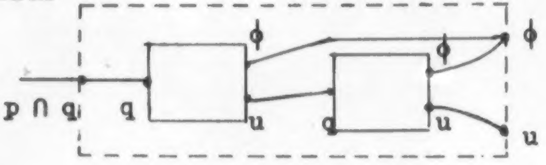


Fig. 24

This corresponds to switches wired in series, as in Fig. 24. Note the strong duality between these two drawings.

Observe, both from the algebra and the drawings, that $p \cup q$ and $p \cap q$ are again switches of the same basic type: they have one input and two output terminals and they are in one of two possible states. Either current will pass out of the off terminal or out of the on terminal. By repeated application of these two constructions plus negation, any expression which can be formed in the two element Boolean algebra can be reproduced as a switching circuit. Thus, for example, statements in elementary logic can be reduced to switching circuits.

Example: Design a circuit so that a light can be turned on by a switch at any one of three doors. Let the switches be denoted by p, q, and r. The overall circuit is to be "on" whenever either of p or q or r is "on," i.e., it has the formula $p \cup q \cup r$. By one of the axioms of a Boolean algebra, this can be written $(p \cup q) \cup r$. The circuit for $p \cup q$ is given in Fig. 23, so $(p \cup q) \cup r$ must be the one shown in Fig. 25.

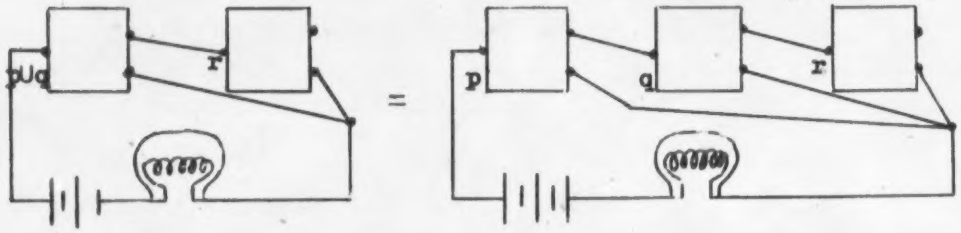


Fig. 25

Example: A light is to indicate when two switches p and q are both in the same position, i.e., the light is to be on if and only if both p and q are on or both are off. Thus, the formula for the circuit is $(p \cap q) \cup (\overline{p} \cap \overline{q})$. The construction for this circuit is shown step by step in Fig. 26. In actual construction, the pairs of switches with the same labels would be ganged switches operating together.

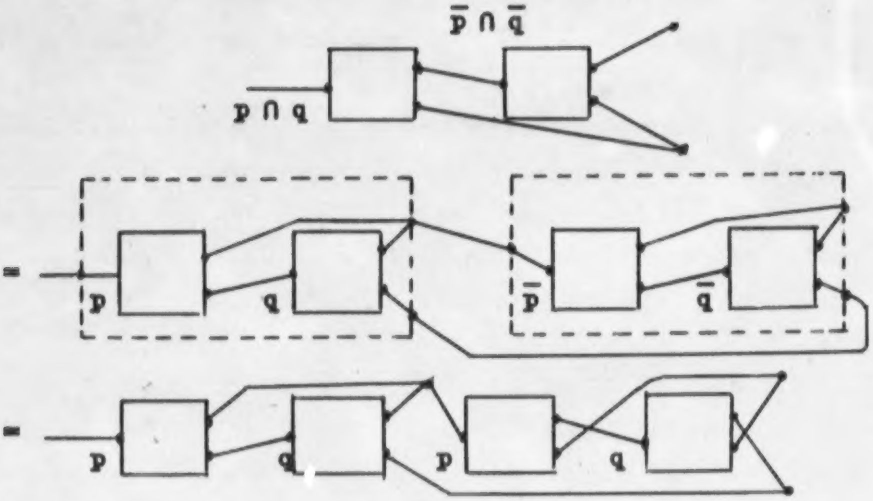


Fig. 26

The merit of making the identification between switching circuits and the two element Boolean algebra is that it is comparatively easy to translate a complicated verbal statement of the conditions to be met into an algebraic formula, then to use the axioms to reduce this expression as much as possible, and then systematically to realize the simplified expression as a circuit.

Example: Suppose that a circuit must be designed to lock the gate of a plant. It is stipulated that the gate shall be open only when one of the following conditions is met:

i. a switch p in the president's office and another, q, at the guard's station are both on;
ii. when p is off, the gate is open if both the guard's switch and one, r, controlled by the security officer are on;
iii. to insure extra protection at night there is a time clock switch which, when off, keeps the gate locked unless all three switches -- the president's, the guard's, and the security officer's -- are on.

Design this circuit.

The circuit must be in one state -- on or off, it doesn't really matter -- if and only if one of the following conditions are met: either

i. $p \cap q$, or ii. $\overline{p} \cap q \cap r$, or iii. $\overline{s} \cap p \cap q \cap r$.

Thus, the overall circuit is

$$(p \cap q) \cup (\overline{p} \cap q \cap r) \cup (\overline{s} \cap p \cap q \cap r).$$

A few manipulations of the axioms shows that this is equivalent to the circuit $q \cap (p \cup r)$. Interpreted verbally, the gate is open if and only if the guard's switch is on and either the president's or the security officer's is also on. In other words, the time clock plays no role and the president and security officer have the same degree of control. The realization of the circuit is shown in Fig. 27.

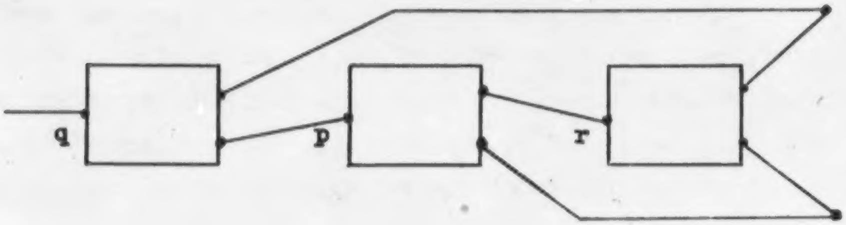


Fig. 27

It should be added that there are a variety of ways of drawing the schematics of such circuits. This style was chosen because it shows very clearly the dual roles of union and intersection (see Figs. 23 and 24).

Problems

1-Draw the circuits for $p \cup \overline{p}$ and $p \cap \overline{p}$ in terms of the single (ganged) switch p. By drawing the two different positions for knife in p, show that the former amounts to a wire and the latter to an open circuit.

2-Prove $(p \cap q) \cup (p \cap q \cap r) \cup (s \cap p \cap q \cap r) = q \cap (p \cup r)$ using the axioms of a Boolean algebra.

3-Suppose a door is locked if p is off, or q is on, or if p is on and q is off. Express this circuit algebraically, simplify, and draw the resulting circuit in terms of the two switches p and q.

*4.6 EXISTENCE OF LINEAR UTILITY FUNCTIONS

Judging by our emphasis on the axiomatization of functions, it is reasonable to conclude that most of the successful axiomatizations related to behavioral problems have assumed that form. However, some important ones exist which are axiomatizations of systems, not functions. One of the simplest to describe, and one for which some background was established in section 3.11, is the axiomatization of a relation over a set of gambles such that a linear utility function exists. You will recall that when we first encountered this problem we presented some of the background and then inquired into the uniqueness of such functions when they do exist. Of course, it is of even more interest to know when they exist, for if we know what conditions the preference relation must meet then we may have some idea whether it is suitable to use linear utility functions to represent preferences.

As background for this discussion, it would be wise to reread section 3.11. There we began with a finite set A of alternatives and from this generated the set G of all gambles based upon A. One way to look at G is as the set of all probability distributions over the elements of A. Another way is to think of G as composed of all elements of A and all elements of the form $a\alpha b$, where $a, b \in G$ and α is a real number, $0 \leq \alpha \leq 1$. Such an element is interpreted as the gamble in which the outcome is a (possibly another gamble) if an event having probability α of occurring actually occurs and b if the event fails to occur.

The elements of G are all possible outcomes which may arise from the basic outcomes A in a situation where there is risk -- where chance events play a role in deciding exactly what outcome will result. It is assumed that any given person will have preferences not only among the alternatives in A but also among all the gambles in G. Let us denote this preference relation by $\gtrsim$, where $a \gtrsim b$ means that he prefers gamble a to gamble b or he is indifferent between them.

For the reasons cited in 3.11, it is of interest to know when a real-valued function u with domain G exists having the two properties:

1-(order preserving) $a \gtrsim b$ if and only if $u(a) \geq u(b)$,

and

2-(linearity) $u(a\alpha b) = \alpha u(a) + (1 - \alpha)u(b)$.

Such a function, when it exists, is known as a linear utility function (for the preference relation $\gtrsim$ on G), and we established that, while not unique, it is determined up to a positive linear transformation.

The problem now is to establish conditions on $\gtrsim$ such that a linear utility function exists. It will be recalled that, to prove the uniqueness result, we found it necessary to derive in section 3.11 a property which $\gtrsim$ would have to be met if a linear utility function exists. It was: if $a \gtrsim b \gtrsim c$, then there exists a probability α such that $a\alpha c \sim b$. It seems plausible that there are other properties which we might establish in much the same way. Some of these might be independent of the ones previously derived; others might be logical consequences. Furthermore, it is plausible -- though not certain -- that if we derive enough of them we will finally be able to show that whenever these are met, then a linear utility function has to exist. Such a set of properties $\gtrsim$ constitutes the axiomatization for which we are looking. As in all such axiomatizations, there is no unique set of axioms which will yield the result we want. Various, apparently quite different, sets will do equally well in the sense that they are all logically equivalent to each other and from any of them existence of a linear utility function can be established. The choice among them is purely psychological. One wants axioms which are fairly simple to understand and which have a certain plausibility. We want them to be in a form that a person will agree (before he knows the theorem) that preferences do (or should) satisfy these conditions. For, by agreeing to them, then he has implicitly agreed that preferences can (or should) be represented by a linear numerical utility function. This is something we cannot expect him to agree to directly, and the whole point of the axiomatization is to transform the problem to a different level where he is more certain how preferences do (or should) behave.

Let us present such a set of seven suitable axioms.

Axiom 1. $\gtrsim$ is a weak ordering of G.

As we pointed out earlier, this axiom must be satisfied by any preference relation which is to be simply described numerically. Intuitively, there seem to be two major doubts about it. First, it implies that strict preferences are transitive: if a is preferred to b and b to c, then a is preferred to c. It seems that we all feel that preferences should be like this, but it is not difficult to devise sets of alternatives which lead people into intransitive traps. Second, it supposes that indifference is also transitive, and that seems questionable. For example, in seasoning food most of us would agree that we are indifferent between a given amount of pepper and that plus one grain more. But if indifference were transitive, we would have to conclude that we are indifferent between any two amounts of pepper, which is silly. Two alternatives seem possible; either we can attempt to change the model quite seriously, for instance by introducing probabilities of preferences, or we can say that a weak order is an approximation to reality -- an approximation which is sometimes pretty good.

Axiom 2. $a \underset{\sim}{>} b \underset{\sim}{>} c$ implies there exists an α, $0 \geq \alpha \geq 1$, such that $a\alpha c \sim b$.

This condition was derived in section 3.11, and we discussed its meaning there. It amounts to saying that there is a continuity of preferences. Again, it is doubtful that it is ever strictly true for preferences, but it seems like a plausible approximation to reality.

Axiom 3. Let α be any number such that $0 < \alpha < 1$, then $a \underset{\sim}{>} b$ if and only if $a\alpha c \underset{\sim}{>} b\alpha c$ for any $c \in G$.

Assuming a linear utility function exists, this property of $\underset{\sim}{>}$ is proved in much the same way as the preceding property. It has a very reasonable meaning: if you prefer a to b and then form the two gambles $a\alpha c$ and $b\alpha c$ (note that the same probability and the same alternative c enters into each gamble), then your preference for a controls your preference between the gambles.

Axiom 4. $a\alpha a \sim a$.
Axiom 5. $a\alpha b \sim b(1 - \alpha)a$.
Axiom 6. $a1b \sim a$.

These three are extremely simple to prove. For example, to show axiom 4,

$$u(a\alpha a) = \alpha u(a) + (1 - \alpha)u(a) = u(a),$$

by linearity of u. But since u is order preserving, this implies that $a\alpha a \sim a$. These are equally easy to interpret. Just think of what $a\alpha a$ means and you see that a person would have to consider it preference-wise indifferent to a. The other two are equally plausible.

Axiom 7. If α and β are not both 0, then

$$a\alpha(b\beta c) = (a \frac{\alpha}{\alpha + \beta - \alpha\beta} b)(\alpha + \beta - \alpha\beta)c.$$

This is arrived at as follows:

$$\begin{aligned}
u[a\alpha(b\beta c)] &= \alpha u(a) + (1 - \alpha)u(b\beta c) \\
&= \alpha u(a) + (1 - \alpha)\beta u(b) + (1 - \alpha)(1 - \beta)u(c) \\
&= (\alpha + \beta - a\beta)[\frac{\alpha}{\alpha + \beta - \alpha\beta} u(a) + (1 - \frac{\alpha}{\alpha + \beta - \alpha\beta})u(b)] \\
&\qquad + (1 - \alpha - \beta + \alpha\beta)u(c) \\
&= (\alpha + \beta - \alpha\beta)u[a \frac{\alpha}{\alpha + \beta - \alpha\beta} b] + [1 -(\alpha + \beta - \alpha\beta)]u(c) \\
&= u[(a \frac{\alpha}{\alpha + \beta - \alpha\beta} b)(\alpha + \beta - \alpha\beta)c].
\end{aligned}$$

Possibly the easiest way to criticize this axiom is first to consider a special case of it in conjunction with axiom 4, namely:

$$(a\alpha b)\beta b \sim a\alpha\beta b.$$

The assumption then is that the two stage gamble on the left, which involves only two ultimate alternatives, is held indifferent to the one stage gamble on the right. First, we observe that the two basic alternatives have the same probability of occurring in both cases. Thus, rationally, it certainly is a reasonable assumption, but it does imply that the person does not receive any pleasure from the gambling itself. Only the final chances over the alternatives count. Possibly, it is reasonable for certain important applications, such as business ones. There, one would hope, only the risks involved should be considered, not whether they are divided into one or two stages.

Even if you feel, as we do, that most people's preferences do not satisfy these axioms, you can still feel that under some conditions it would be desirable for a decision maker to satisfy them. If so, then you are saying that ideally his preferences should be represented by a linear utility function, for it can be shown (we will not do so here, but see problem 3) that any preference relation $\gtrsim$ on a set G of gambles which satisfies these seven axioms can be represented by a linear utility function. This famous theorem was first proved by von Neumann and Morgenstern in the second edition of The Theory of Games and Economic Behavior.

As we said, a number of other equivalent axioms systems can be found in the literature. The only merit that really can be claimed for one over another is that the axioms of one seem more intuitively reasonable. There are numerous counter intuitive examples available where one or another axiom appears to be untenable.

One of the main uses of this theory is to justify introducing numerical payoffs into the theory of games. In section 2.6 we described what is meant by the rules of the game. Among other things, there was the game tree which described the pattern of decisions for the various players. Assigned to the end points of the tree were certain outcomes, for the end point of the tree describes a unique path of decisions through the tree. Now, if it is assumed that each of the players has a preference pattern among the possible outcomes and gambles involving these outcomes which satisfy the seven axioms above, then we know that for each player each outcome can be represented by a numerical utility. Thus, we can replace the assignment of outcomes to the end points by the assignment of the corresponding utilities. This new structure, which is the same as the rules of the game except that there are numbers (utilities) instead of outcomes, is known as the extensive form of a game.

*Problems

1-Using some of the seven axioms, show $a0b \sim b$.

2-Derive the property expressed in axiom 5 from the assumption a linear utility function exists. Discuss its plausibility.

3-Choose any a, b $\in$ G such that $a > b$. Define $u(a) = 1$ and

$u(b) = 0$. For any other $c \in G$, how would you use axiom 2 to determine the value $u(c)$? (In the existence proof, this is how u is actually constructed; the other axioms are then utilized to show that it is both order preserving and linear.)